Art of Modeling in Contact Mechanics

Michel Raous
Aix-Marseille Univ, CNRS, Centrale Marseille, LMA, France
raous@lma.cnrs-mrs.fr *

Abstract In this chapter, we will first address general issues of the art and craft of modeling - contents, concepts, methodology. Then, we will focus on modeling in contact mechanics, which will give the opportunity to discuss these issues in connection with non-smooth problems. It will be shown that the non-smooth character of the contact laws raises difficulties and specificities at every step of the modeling process. A wide overview will be given on the art of modeling in contact mechanics under its various aspects: contact laws, their mechanical basics, various scales, underlying concepts, mathematical analysis, solvers, identification of the constitutive parameters and validation of the models. Every point will be illustrated by one or several examples.

1 Modeling: the bases

It would be ambitious to try to give a general definition of the concept either of a model itself or of model processing. Modeling relates to the general process of production of scientific knowledge and also to the scientific method itself. It could be deductive (from the general to the particular, as privileged by Aristotle) or inductive (making sense of a corpus of raw data). Descartes (38) saw in the scientific method an approach to be followed step by step to get to a truth. Modeling can be effectively regarded as a scientific method that proceeds step by step, but its objective is more modest: to give sense of an observation or an experiment, and above all to predict behaviors within the context of specific assumptions. This concept of "proceeding step by step" is fundamental in modeling.

In this first section, we will examine the notion of model in the general context of mechanical systems. The main features of these models are their

*The author is grateful to Prof. Marius Cocou for his very constructive comments on the mathematical aspects and Prof. Alfredo Soldati for our very interesting discussion on the general topic.

contents, their purposes, the way to build them, the way to use them, the expectations and challenges.

Some of the reflections presented in this section find inspiration in the book entitled Mathematical Modeling Techniques written by Aris Rutherford, from University of Caltech, in the 80s (108).

1.1 The objectives

The objective of modeling is twofold: explaining and understanding, giving sense to experimental observations on the one hand, predicting behaviors on the other one.

Understanding.
The models we are talking about all start from experimental observations to which we want to give sense (deductive approach) and that we wish to explain, understand and describe using the basic laws of mechanics and adding "properly" chosen theoretical ingredients. The term "properly" is fundamental and could be very constraining.

With the goal of understanding, the approach consists in gathering the inherent elements based on concepts necessary to build the systems of equations to be solved. Behind that, there are some questions that should be answered: do the solutions exist, are these solutions unique, are they stable, are there methods (existing or to be developed) for solving them? In other words, a model is part of a theoretical mechanical and mathematical framework, and it will be essential to make sure that this framework has been properly chosen, that there is consistency between the equations and the framework, and that the possible limitations can be identified.

The main ingredients when building models in Mechanics are based on the fundamental laws of mechanics and thermodynamics, as well as on those of physics, electricity and chemistry, and they need either the use or the development of mathematical tools. In the following we will present how, in some cases, modeling requires new mathematical developments, this being particularly true for contact problems. The problems studied today are increasingly complex, due to interactions between phenomena of different natures, and multiphysic models can be considered.

Using or developing ingredients that have a meaning with respect to some concepts to construct a scientifically coherent model is an essential feature of the art of modeling. This refers to the objectives of "understanding" and "giving sense" in modeling.

Before going any further into the analysis of the modeling processes, let us make mention of other types of approaches. The notion of modeling that we present in the following is actually restrictive, and there are branches of the modeling that differ markedly in their concept. Let us present two of them.

Note 1. When we associate modeling to "equations", we cut off other approaches which are directly based on experiments without theoretical link or at least without a formal theoretical link. Let us give an example of experience-based modeling by referring to the work of the Catalan architect Gaudi, known, among others things, for the basilica church "La Sagrada Familia" in Barcelona. For the design of its arches, Gaudi developed the reversed chain model. He created mockups, using chains simulating the arches weighted with small bags to simulate the bearing forces applied by the supporting pillar on the arch. Then, a mirror placed below gave the reversed image of the structure and showed the final shape of the real arch design (see Fig. 1). He also used small hanging bags (full of salt or flour) for the representation of the shapes and the volumes of the domes, the cupolas, the arches and the pillars. The modeling process uses directly measurements of the mechanical behavior of the mockup without setting a theoretical framework as we will do in the following. However Gaudi's architectural pieces of art brilliantly illustrate the efficiency of the reversed hanging chain model and of this kind of modeling approach.

Figure 1. Reversed chain model and cathedral church La Sagrada Familia

Note 2. Another class of modeling that we will overlook in this presentation is the one based on statistical considerations. In that case, from a (if possible large) number of experiments or random draws, we build a predictive behavior or determine a solution to the problem using sophisticated statistical tools. The Monte Carlo methods have been the basis for this kind of approach. The Monte Carlo method was invented by Metropolis-Ulam

(73). This class of methods is based on random drawings chosen according to suitable probability distributions. Let us give an example. The acoustic field radiated by a source in a volume of any shape may be determined without solving the wave equation but by using a ray theory: the acoustic field is then evaluated as the superposition of the trajectories of sound shots fired from the source in random directions, the absorption by the walls being taken into account in the reflection rule. This approach works quite well as long as the edge diffraction phenomena are not too important. Of course, the higher the number of shots is, the better the approximation is.

Examples of various methods employing statistical and probability tools in the modeling process itself include variance reduction, polynomial chaos, kriging (used in geostatistics, meteorology, environmental sciences and electromagnetism), as well as fractal analysis (see Cherepanov-Balakin (18)). The latter, more recent, is widely used in many fields. Among them can be cited characterization and analysis of rough surfaces, dynamic cracking and elasticity of polymers or rubbers (18), analysis or monitoring of industrial processes, fault detection in machines, etc.

Predicting.

Beyond providing the satisfaction of a better understanding, the main purpose of modeling is to predict, that is to say, replace experimentation by simulation to predict the behavior of a mechanical system submitted to new loads, i.e., different from those used for building and validating the model. Replacing the experiment by a simulation via computation or analytical analysis is a crucial challenge, for two reasons at least. In the first place, replacing a time-consuming and costly experiment by some calculations will be comfortable and economical (although it can also be very expensive at times). Secondly, calculations are indispensable when experiments are not feasible or very difficult to conduct, as in cases such as accidental events or natural disasters (a tsunami, the collapse of a dam or a pipe in an irradiated environment, a nuclear explosion, the movement of a cyclone, etc.). Weather forecasting is a daily example of modeling. In some cases, experiments are performed using mockups but the scale effects remain a problem which should be considered with great caution.

It is to be understood that, for a reliable behavior prediction, the challenges of modeling are the validation of the model, the study of its reliability and the identification of its domains of validity.

The objective "understanding" is the basis for the objective "predicting", and it has an even more fundamental purpose. On the other hand, the economic stakes involved in "predicting" are enormous: a validated model will drastically reduce the development period and the development cost for a new product (in the broad sense of the term).

Modeling is the alternative to a trial-error experimental process, which consists in the construction of experimental protocols conducted by testing the influence of the parameters considered as significant. This approach will be possible for systems exhibiting a linear behavior (which of course will have to be verified) but will soon become intractable in case of nonlinear behaviors, due to the required number of experiments and given the time for and the cost of making each specimen and conducting each experiment. Developing a good model will represent an important challenge.

A validated model will permit the optimization of a mechanical system. It will be an efficient tool in the hands of the engineer who will be able to test the influence of every parameter rapidly. A model is not a universal tool that would replace the scientist. It is only a sophisticated tool that will help the designer or the researcher during the process of creating or optimizing a new product. Their skills in developing, selecting and using the models are indeed of central importance in the art of modeling.

Let us now get an overview on the various steps of the construction of a model:

- the founding step is the choice of the equations chosen to describe the phenomenon which has to be modeled: some assumptions have to be made, the main characteristics to take into account have to be identified, the scale of the analysis has to be chosen, etc.,
- then comes the formulation step, with the choice of the formulation framework,
- the mathematical analysis of the problem should give information about the existence, uniqueness and stability of solutions, and more generally information about the conditioning of the problem to be solved,
- then comes the step of solving the problem which is obviously a key step which leads to numerous developments and may give rise to very difficult problems; this should be completed on the one hand by the numerical analysis of the approximate problem, of the discrete problem and of the algorithms and on the other hand by the validation of the numerical method,
- once the numerical tool is ready, the basic constitutive parameter have to be identified using the model to simulate a test experiment; the sensitivity of the solutions to some changes of the parameter values should also be tested,
- and finally the last step is the validation of the model by simulating various experiments and evaluating the error between the theoretical and the experimental results.

Some of these steps can be more or less important and more or less difficult depending on the cases, but it will be important and useful to keep all of them in mind.

1.2 Construction of a model

The art of the modeler will be about highlighting the key points and identifying the nature of the significant phenomena and their contributions in various experiments. The basic step in building a model is the choice of the ingredients to use and of the assumptions to make. It will be essential to bear in mind that a model is composed of "equations" and "assumptions." These underlying assumptions will be the bases of the validity domain of the model and they have to be clearly defined and remembered. This may seem obvious but transgressions are common: for example, using numerical integration methods based on limited developments for computing solutions that are not differentiable would be incorrect.

Concerning the assumptions, there are also some subtleties, such as the difference between simplification and neglect. The former permits to solve the problem but is not necessarily legitimate (and that should be studied) while the latter consists in neglecting a secondary aspect of the problem (and that is a justified assumption).

We can try to list and classify by nature some of these choices and assumptions that will be made during the construction of the model and of the systems of equations to solve, in order to correctly simulate the experiment.

- The physics that should be taken into account. The first step is to identify the nature of the significant phenomena which have to be considered: mechanical, chemical, thermal, electrical effects or other effects. This defines the degree of complexity of the model. Multiphysic modeling is increasingly used today. This step is very important because neglecting a significant effect (incomplete model) would have as harmful effects as taking into account a side effect (unnecessarily complicated model).
- The "equations" and their framework. The "equations" will be written in the general framework of the fundamental principles of mechanics and thermodynamics. Their implementation gives rise to differences in points of view, equations of motion or energy principles, partial differential equations or minimum problems, etc. The degree of generality or abstraction of the framework will define the degree of generality of the model. We will speak of classes of models when this degree of generality is great. This will be illustrated in Section 2.5 by a work where we introduce a unified model for adhesive interfaces based on

considerations such as the concept of generalized standard materials extended to interfaces.

- The role of time. Time plays a special role in the development of a model. Several questions arise and have to be answered. The first question is whether a dynamic study should be conducted, if the effects of inertia are to be taken into account, or a quasi-static approach is sufficient, if masses are small enough or phenomena slow enough.

  Another question concerns the possible influence of the velocities on the nature of the behaviors and if viscous dissipation should be considered.

  A third question is whether the effects of variations in time of the parameters should be taken into account, either directly (effect of aging for example) or indirectly, these parameters depending on some of the variables such that displacements or velocities.

  When the phenomenon depends on the loading path (plasticity, damage, etc.), then we are in a rather special case where time is replaced by a dimensionless variable characterizing the loading path.
- Choice of the scale. Another specific issue when building a model is the choice of the scale to conduct the analysis. It could be only the choice of a given scale or that of the combination of several scales (multiscale model). With the phenomenal increase in computer power, multi-scale approaches are experiencing spectacular growth.

1.3 Choosing a formulation

Choosing a formulation relates several levels. First, we can choose a discrete or a continuous formulation. Secondly, it has to be noted that the choice of the variables is directly connected to the form of the formulation (primal, dual or mixed formulation).

A discrete formulation consisting in assembling masses, springs and various elements in general has as objective to "understand". A discrete formulation can be very sophisticated, including nonlinear or specific items. Such example is presented in Section 1.9. The target to "simulate" and to "predict" the behavior of complex structures rather goes through continuous formulations associated with finite element methods which may work for any geometry. But many variants are possible and the combination of discrete and continuous formulations is often useful for complex assemblies.

The mathematical framework is closely relates to the formulation. To take some examples in continuum mechanics, the general framework for smooth problems will be that of functional analysis and Sobolev spaces. In some cases, such as contact problems discussed later the convenient framework will be the ones of convex and non-convex analysis. As for the variational formulations, we will distinguish among primal (displacement or velocity formulation), dual (stress formulations) or mixed formulations, each having its own advantages and disadvantages depending on the situation. In the case of non-smooth problems with respect to time, a convenient framework will be that of distributions and differential measures.

For non-smooth problems, the formulations and their mathematical framework may be drastically simplified by using regularizations. However, these processes may create a problem markedly different from the initial problem, which can have an impact on the validity of the model. A special section in the following will be devoted to that important issue.

1.4 The framework and the mathematical analysis

Although often considered as unnecessary or irrelevant, or else as an exercise for mathematicians, mathematical analysis has a leading role in the art of modeling. For smooth problems (with respect to time as well as space variables) the mathematical framework is very classical and does not need special care. However, it can be noted that only the writing of boundary conditions for partial differential equations uses the concept of trace in Sobolev spaces.

Note that whole areas of mathematics have been developed from problems in mechanics: some aspects of functional analysis, convex analysis, the notion of Γ-convergence, etc. (see Dautray-Lions (33)). Mechanical problems, especially non-smooth problems, have promoted very high level mathematical developments which are still ongoing. Unfortunately, despite giving an adequate framework, they sometimes lead to results that cannot be quantitatively used directly by engineers. However, they always help in the understanding of both the model and the computational behaviors. The theoretical framework and the mathematical analysis are very constructive in setting the problems properly, understanding the main theoretical properties and developing convenient numerical methods for the resolution. We want to stress that using the right mathematical context adapted to the ingredients (equations) put into the model is a key point in the art of modeling, the only way to find and to characterize the right solutions, including sometimes some unexpected ones.

We will illustrate that in the following. Contact mechanics will be a good context to discuss modeling in these various aspects. Although the laws seem very simple (Coulomb's law dates back to the 18th century), everything is very complicated because they relate to non-smooth mechanics: threshold laws, the graphs of the contact laws are multivalued mappings and not functions, etc.

1.5 Resolution

It is obviously a key step in the art of modeling. The resolution could be analytical: it is then limited to elementary geometries, but it simplifies parametric studies and helps to identify the essential phenomena more easily. More generally, analytical approaches are useful to "understand" and hence they must not be neglected in modeling.

But today, the methods of resolution are mostly numerical and the great increase in computer power has paved the way not only for new methods but also for new ways of thinking about modeling approaches. There is now in the art of modeling a close link between resources and computing power and design patterns.

We would like to point out a downside risk associated with this increasing computational power. It is the risk of replacing "thinking" by "large number of degrees of freedom". This is undoubtedly one of the main common pitfalls of modeling. A lot of computations using millions of Degrees Of Freedom (DOF) will very often provide less on the knowledge and the understanding of a problem than a well thought out model using a small number of DOFs or a discrete mechanical model.

The range of existing methods is very wide and in constant development. Without trying to be exhaustive, one may mention the finite element methods (FEM) and their variants ($(\mathrm{FEM})^2$, subdomains, multigrid methods, diffuse elements, etc.), finite volume methods, integral methods, spectral methods, etc.

It is worth stressing several points about numerical methods and computations.

First, it will be important to keep in mind that a numerical method must be consistent with the assumptions and with the mathematical framework.

Another extremely important point is the validation of the numerical method itself (see Section 4). This must be done using standard solutions that can be analytical or qualified by other methods (benchmarks). This is not trivial matter and it is sometimes possible to find numerical methods validated with an experiment - which is absurd. What is more, this step of validation of the numerical method will generally be an opportunity to evaluate and control the numerical errors (distinguishing between discretization errors and calculation errors), check convergence conditions, properly select calculation parameters, etc.

The numerical analysis of the approached problems and of the discrete problems is often overlooked because it is often difficult. However, it will provide important information about the stability of the algorithms (and possibly provide the stability conditions to be verified), on convergence, error estimates and other properties that will be valuable during the numerical implementation.

1.6 Identification of the parameters

The parameter identification step should be distinguished from the model validation step - which we will be discussed in the following. This essential step comes after the model construction step, and consists in finding a solution to the optimization problem: finding the parameter values which minimize the gap (the error) between one experiment and the results given by the model simulation.

Generally, the identification of constitutive parameters is conducted with experiments that are either independent of the problem addressed or closely linked to it. For example, when modeling the behavior of a structure, the mechanical properties of the materials are usually determined with experiments on specimens, thus independently of the system studied.

On the contrary, in other cases, these characteristics depend so much on the environment of the system in real situation that it becomes necessary to use other processes. This will generally be the case for interface models, as we will see later.

1.7 Validation of the model

This is of course the key step in modeling since it will allow us to give reliable behavior predictions when using the model. Validation is part of the most important steps in modeling; it will define the quality of the model, specifies the validity domain and confirm its ability to be used by the designer. It is essential and requires special care (see Section 4).

Validation is carried out on one or more experiments by evaluating the error between the results of the experiments and those of the simulations, using the values of the constitutive parameters obtained by the identification process. Let us note in passing that the norm selected to evaluate this error is of great importance since a norm L^2 (squared error), which inherently carries an energy sense, will smooth the gaps, while a norm L^∞ (sup norm) will measure the maximum deviation which can in some cases be the decisive dimensioning factor.

Note that the experimental conditions should verify the assumptions made initially for the construction of the model (load amplitudes, speed, etc.).

At this level, adjustments to the model will be performed by iterations between the model prediction and the experimental results. Two approaches may be followed: either improving the model if the gap between theory and experiment is not small enough or checking whether it is possible to simplify the model while keeping an appropriate gap value. The second approach is often overlooked.

1.8 To conclude this introductory chapter

In this chapter, we have addressed many aspects of modeling. As a complement, we would like to stress two key points in the art of modeling.

This first key point is: "Make as simple as possible and as complex as necessary, no more, no less!".

We will illustrate this with the example of the famous work of Pablo Picasso, "Bulls" (Picasso, 1945) presented on Fig. 2. 'Bull' is a suite of eleven lithographs that have become a master class in how to develop an artwork from the academic to the abstract. In this series of images, Picasso visually dissects the image of a bull to discover its essential presence through a progressive analysis of its form. Each plate is a successive stage in an investigation to find the absolute "spirit" of the beast. In this work, Picasso progressively simplifies the shape of the bull and keeps the quintessence of the draw. In the final print of the series, Picasso reduces the bull to a simple outline which is so carefully considered through the progressive development of each image that it captures the absolute essence of the creature in as concise an image as possible. A critic made that comment: "He has ended up where he should have started! He had gone in successive stages through all the other bulls. When you look at that line you cannot imagine how much work it involved. He had in mind to retrieve the bull's constituent parts, his dream bull - bred of pure lines - an elemental, disembodied, quintessential bullishness". The result is the Platonic idea of a bull. The reverse process

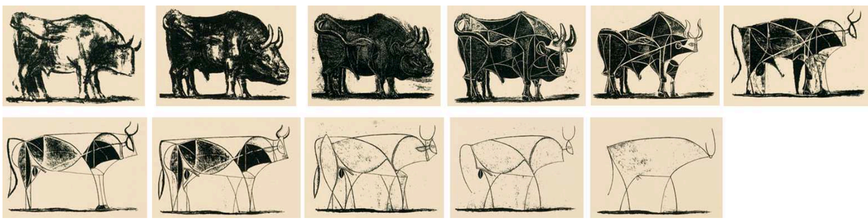


Figure 2. The process followed by Picasso in the eleven lithographs "Bull, 1945"

to get back to the initial complete image of the bull is the gain of knowledge. This development of knowledge to obtain the ideal model from a reality is the basis of the art of modeling. The process of Picasso is in fact exactly our only hope to gain knowledge.

In the same state of mind, we may also cite Fernando Pessoa: "Thought must start from the irreducible".

The art of modeling consists precisely in building the simplest model containing the key characteristics of the phenomena and permitting its suitable and validated simulation. Making the model as simple as possible and as complex as necessary is really fundamental in the art of modeling: making it too simple would be insufficient and making it too complex would be useless and costly.

A second key point in the art of modeling is: you should not fall into the trap of habit. One of the pitfalls the modeler must avoid when faced with a new problem is the temptation to do "what he does best "rather than "what he should do", i.e., fitting the problem within the framework of his own mechanical engineering culture, of his usual tools (even following necessary new developments) instead of using a totally different approach, which should be more convenient, but not familiar to him. It is not easy to resist a natural temptation and it is thus often difficult to avoid the trap of habit, as it requires a sound knowledge of models and modeling, when models are becoming increasingly sophisticated, and it is difficult to be an all-rounder. It will be more constructive to work within multiple skill groups or to liaise with a network of specialists, while having developed a sufficiently broad knowledge, from experimentation to numerical methods and mathematics, to be able to dialogue with those specialists. This variety of knowledge in modeling is well known to industrialists, who sometimes present the same problem to several academic research laboratories that differ in their approaches.

To conclude on this chapter, we can say that, besides the broad spectrum of his scientific qualities, his in-depth knowledge, the thoroughness of his analysis, intuition will be a key asset for the modeler.

We will now illustrate some features of this section with an example from mechanics of materials, the cyclic behavior of a polyurethane foam, before getting to the art of modeling in contact mechanics in the next sections.

1.9 Step by step construction of a discrete model - Example of the modeling of the cyclic behavior of a polymeric foam

This problem, studied in the PhD thesis of Giampiero Pampolini ((82) (81) (36)), will be used to illustrate three points discussed in this section.

- "Discrete model and analogical models" (springs, dashpots, etc.) help to "understand" and to identify the driving phenomena. We use a discrete model composed by assembling non-linear springs, dashpots and other convenient elements. The goal is not to simulate the behavior of a whole structure but to identify and to model the main phenomena occurring in this kind of material during cyclic loadings.

- "From simplicity to complexity": the model is developed step by step. This second point relates to the idea to put into a model what is necessary, no more, no less. We start with simple elements and we progressively add other necessary elements.
- "Identification of the constitutive parameters" and "validation of the model" which have to be clearly distinguished

This is a discrete model used to "understand" and which is "as simple as possible and as complicated as necessary". All details can be found in Del Piero-Pampolini-Raous (82) (81) (36).

Main characteristics of the behavior: the experiments.
When a sample of an open-cell polymeric foam is submitted to quasi static compression, localization of the deformation can be observed with the occurrence of bands orthogonal to the loading direction (see Fig. 3).

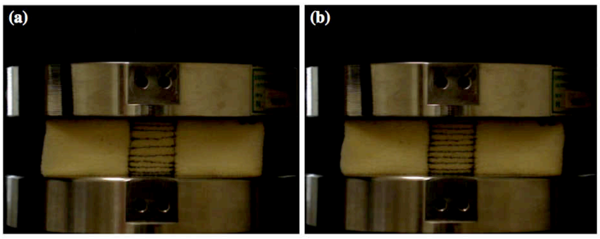


Figure 3. Localization of deformation during compression

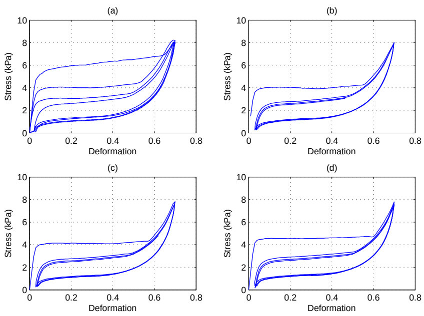


Figure 4. Recovering (a) Virgin material, Resting period : (b) 16 hours ; (c) 52 hours ; (d) 33 days.

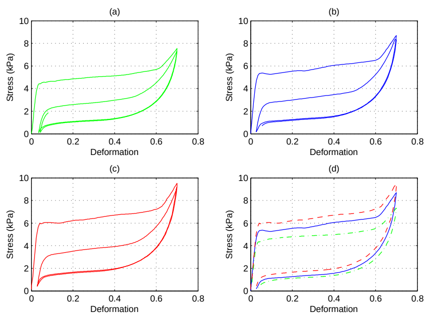


Figure 5. Loading velocity influence: (a) 0.1 mm/min ; (b) 5 mm/min ; (c) 100 mm/min ; (d) Comparison of the 3 results.

When a cyclic compression is applied, the following characteristics of the stress/strain cycles are noted (see Fig. 4 and Fig. 5):

- specific shape of the cycles,
- first cycle different from the next ones,
- influence of the loading velocity,
- influence of the resting periods on the behavior,
- analogy with Mullins effect for elastomers (strain softening).

Discrete problem and nonlinear elasticity.
A discrete model, an assembly of springs, dashpots and other convenient elements, is considered. The specimen will be considered as discretized into a finite number of layers (Fig. 6). The art of modeling will consist in choosing the good characteristics of the various constitutive elements of the chains. We will first address the localization phenomenon observed during compression. For that, we consider that the springs exhibit a nonlinear elasticity based on the non-convex strain energy density w presented in Fig. 6 (details can be found in (81) (36)). Details on the analysis and on the computation of the response can be found in (82). The existence of two phases is characterized: one phase corresponding to the virgin cells and the other corresponding to the totally squeezed cells.

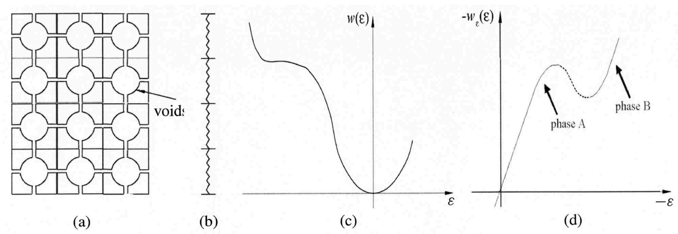


Figure 6. Representation of the cell lyers (a) by a chain of springs (b) with a non convex strain energy (c) which gives the behavior law (d) for each spring.

A good agreement is observed between the model and the experiment for the first compression step, but, either when unloading or when various loading amplitudes are considered, there is a big gap between theory and experiment results (even qualitatively), as shown in Fig. 7.

Viscosity effects are added.
The model has to be completed by taking into account other phenomena. We will now introduce the viscosity effects. The idea is to incorporate what is necessary gradually into the model to describe the experimental

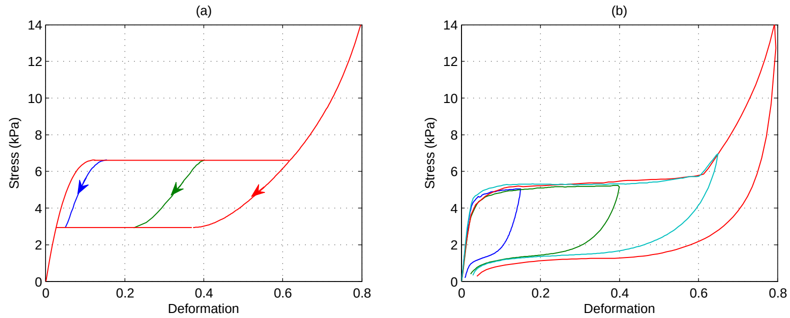


Figure 7. Comparison between the nonlinear elasticity model (a) and experiment (b) for different loading amplitudes.

behavior. We introduce a Zener model with Maxwell viscosity. This model may encounter both the loading rate dependence (classical viscosity effect during the cycles) and a long period dependence (recovery of the initial properties after resting periods which is a relaxation phenomenon). For that, we introduce several relaxation times with different time scales in the viscosity model. In Fig. 8, it can be observed that a good agreement between theory (full line) and experiment (dotted line) is now obtained regarding the shape of the cycles. Nevertheless, even if the first cycle is very well described, it remains that the gap between the first cycle and the next ones is not described.

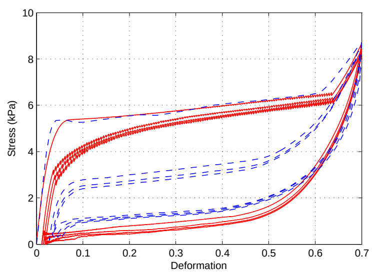


Figure 8. Theory/experiment comparison when using the nonlinear elasticity model with viscosity

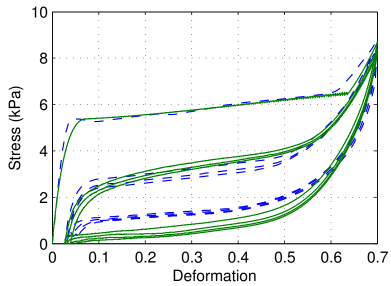


Figure 9. Theory/experiment comparison when using nonlinear elasticity, viscosity and damage.

Damage is added.
Once again, a new effect has to be taken into account and we introduce damage in the model. Now, as shown in Fig. 9 a very good agreement is obtained between the model simulation (full line) and the experimental results (dotted line). The complete model has been constructed step by step, progressively adding new concepts that are necessary to get an accurate description of the behaviors observed during the experiments.

A model to "understand" Let us note that this discrete model is very helpful to understand the behavior of polymeric foam. "Understanding" was presented as the first goal in the art of modeling in Section 1. We explain that this cyclic behavior is due to nonlinear elasticity, viscosity and damage. The model shows clearly the phase-change mechanism occurring in the polymeric foam during the loading, which gives the shape of the stress/strain curve. With a simulation conducted with a chain of 120 elements, the panel of colors (Fig. 10) gives the deformation levels in the column (this is a one dimensional model) corresponding to the different points on the stress/strain curve: A, D, E and G are single phase configurations, while B, C, F and H are two phase configurations (in B, C and F the two phases are block separate). The points C, E and H show situations with different stresses for the same deformation. This is due to the dependence of the solution on the loading path. A good understanding of the phenomenon is achieved.

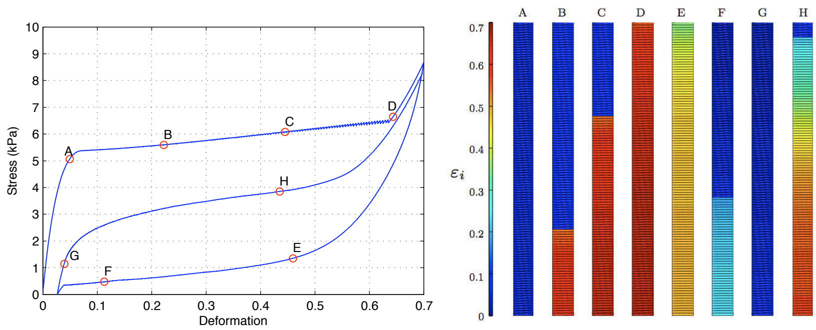


Figure 10. Evolution of the deformation condition (phase changes) during a cycle (the color scale for the deformations is given on the far left)

How to validate the model ?
The good theory-experiment agreement shown in Fig. 9 is the result of the process of identification of the constitutive parameters. The identification was a quite difficult process and details of the methodology will be presented

in Section 4.4 and in Pampolini-Raous (82). At this point (construction of the model and identification of the parameters), we only showed that there is at least one set of parameter values that permits to fit the model with one experiment. The model is not yet validated and not ready to be used for "prediction", the other goal of the art of modeling. In order to validate the model, we will use it, with the parameters evaluated during the identification process, to simulate other kinds of loadings. This time, a good agreement between theoretical predictions and experimental results will confirm the validation of the model. This will be shown at Section 4.4.

1.10 Conclusions about the art of modeling for this first example

This discrete model has helped to understand what the main mechanical phenomena involved in the cyclic behavior of polymeric foam are and, what is more, has proposed some forms for the various ingredients that have to be used:

- nonlinear elasticity with non-convex strain energy (a form is proposed),
- viscoelasticity with at least two time scales (cycle period and duration of the resting periods); a Zener-Maxwell model could be convenient,
- damage.

What is more, this work on this discrete model gives an order of magnitude of the various constitutive parameters for the mechanical model.

Let us note that the model has been constructed step by step in order to make "as simple as possible but as complex as necessary". The way to conduct "identification and validation" will be presented at Section 4.4.

The next step would be now to develop a continuous formulation of the problem and to use finite element methods to simulate the behavior of a polymeric foam structure in engineering systems, such as energy absorbers, seat cushions, packaging materials and lightweight composite sandwich structures, etc. That is not simple but the present model helps to understand what the main ingredients are that should be put into such a model.

2 Building models in contact mechanics

We will now address the art of modeling in contact mechanics. As already mentioned, contact mechanics is a field of interest for discussing the art of modeling, since many of the issues addressed in the previous section will raise difficult questions.

In this section, we will focus on the step "building the model" - which is the most mechanical part. We give a panorama of various contact laws (unilateral contact, friction, adhesion, wear, etc.) and present the diversity of the applications in various fields. The different scales that can be used are presented: we give some elements and references on multiscale analysis which is expanding strongly. We show how a model is built on the basis of concepts. We stress the advantages versus the dangers of regularization.

This is illustrated with two examples:

- a law coupling unilateral contact, friction and adhesion (the RCCM model) to illustrate the importance of concepts as the basis of a new contact model (thermodynamic analysis) and the mathematical complications; this is applied to an engineering problem: how to improve the resistance of a composite material to the crack propagation by controlling and optimizing the behavior of the fiber/matrix interface,
- a unified model coupling contact, friction and adhesion to illustrate the inductive/deductive process when one elaborates a model on the basis of fundamental mechanics (generalized standard materials).

2.1 Basic interface models

Let us present a panorama of the main interface laws corresponding to the main usual behaviors that we want to describe: non penetration into the obstacle, friction, adhesion, healing adhesion, wear, etc. Details can be found in Raous (95) (103) (97) and Sauer (109).

Unilateral conditions.
The basis is to write that the solid cannot penetrate into the obstacle supposed to be rigid. It can be generalized to the contact between two deformable solids. Let us consider a rigid obstacle, the solid occupying a domain Ω, let Γ_C the part of the boundary Γ initially in contact with the obstacle, u_c the displacement (trace on Γ_C of the displacement u defined on Ω), R the contact force and n the outward normal vector to Γ_C. Using the following partition between normal and tangential components, the non-penetration condition is described by the following Signorini conditions which constitute a complementarity formulation. The graph of this behavior law is given in Fig.11. This is not a function but a multivalued mapping. It is here that difficulties begin!

$$u_c = u_n n + u_t \qquad R = R_n n + R_t$$

$$
\begin{aligned}
u_n &\leq 0 \\
R_n &\leq 0 \\
u_n R_n &= 0
\end{aligned}
\tag{1}
$$

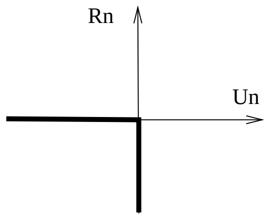


Figure 11. Unilateral conditions : the Signorini conditions

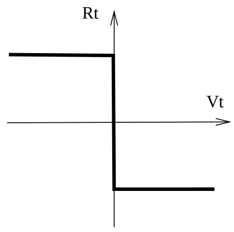


Figure 12. Friction law: the Coulomb law

Friction: Coulomb - Amontons friction law. (see (32))
The basic law of friction is the Coulomb law (1821) which was developed on the basis of experimental observations with moderate amplitude forces. First, it is a law with a threshold, no movement occurs unless the tangential force reaches a certain threshold, secondly, this threshold is directly proportional to the normal force and thirdly, the movement occurs in the direction opposite to that of the applied tangential force. This is expressed by the following law:

$$
\begin{array}{llllll}
 & \|R_t\| & \leq & \mu|R_n| & & \qquad (2)\\
\text{if} & \|R_t\| & < & \mu|R_n| & \text{then} \quad \dot{u}_t = 0 & \qquad (3)\\
\text{if} & \|R_t\| & < & \mu|R_n| & \text{then} \quad \dot{u}_t = -\lambda R_t \quad \text{with } \lambda \geq 0 & \qquad (4)
\end{array}
$$

This law seems to be very simple; only one parameter, the friction coefficient μ, is involved. However, at the same time, it is very complicated because it is not a function but a multivalued mapping (see Fig. 12), as for the unilateral conditions. It will drastically increase the difficulties when writing the mathematical formulations and developing the numerical methods. We should stress that this non smooth character is precisely the fundamental richness of this law; it will make it possible to model specific behaviors such as occurrence of instabilities (squeal, stick-slip, etc.), existence of multiple solutions, etc. Therefore, it will be important to conveniently deal with this non smooth character; we are in the context of non-smooth mechanics.

The configuration space is given by the Coulomb cone in the force space (see Fig. 13). When the representative point is inside the cone, no movement occurs, when it reaches the boundary, sliding occurs with the direction

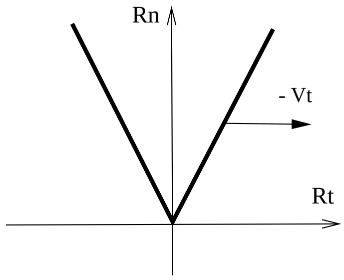


Figure 13. The Coulomb cone

of the tangential force (and not orthogonally to the boundary of the domain, as it is the case for example in classical plasticity). Thus, we should note that there is no normality rule (the friction law is non-associated) and this will lead to extra difficulties for the formulation (no minimum principle, as we will see in Section 3.3). As announced in Section 1.3, various formulations can be given and modeling consists in selecting one among them. Other formulations of the Coulomb law can be given. Let us here briefly give some of them (details can be found in Moreau (76)):

- maximum dissipation principle

$$R_t \in C \qquad \forall S_t \in C \qquad (S_t - R_t)\dot{u}_t \geq 0 \tag{5}$$

  where, in 2D $\qquad C = [-\mu R_n, +\mu R_n]$
- subdifferential formulation

$$-\dot{u}_t \in \partial I_{C_t(R_n)}(R_t) \tag{6}$$

  where $\partial I_{C_t(R_n)}(R_t)$ is the subdifferential of the indicator function of $C_t(R_n)$ with $\qquad C_t(R_n) = \{P \text{ such that } |P| \leq -\mu R_n\}$
- dual subdifferential formulation

$$R_t \in \partial \phi_{R_n}(-\dot{u}_t) \tag{7}$$

  with $\qquad \phi_{R_n}(-\dot{u}_t) = \mu R_n \|\dot{u}_t\|$

Possible regularizations.
The comments presented in this section are of great importance for the art of modeling. Regularization is a natural tendency when we have to deal with a non-smooth problem. It allows us to replace the initial problem by

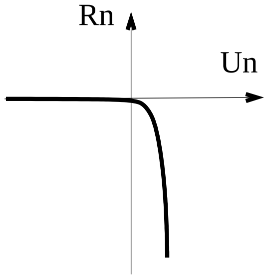


Figure 14. Compliance or penalization of the Signorini problem

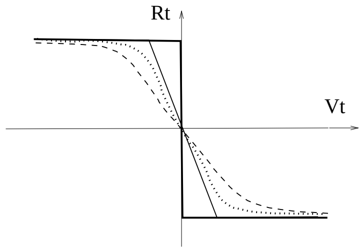


Figure 15. Regularizations for the Coulomb friction Hyperbolic tangent (dotted line), square root (dashed line), polynomial (solid line)

a much simpler one, which is also simpler to solve. We want to stress a very important point: a regularized problem is different from the initial problem and could even model a very different mechanical phenomenon. The modeler should be very aware of that. Contact problems may illustrate that point very well.

For contact problems, regularization will consist in replacing the multi-valued mappings by functions in order to obtain a smooth (but non-linear) problem.

- Regularization of the unilateral conditions: compliance or penalization of the contact laws. As shown in Fig. 14, the contact behavior is now characterized by a (non-linear) function. When penetration into the obstacle occurs, a strong force is introduced so as to push the deformable solid out of the obstacle. A classical choice is the following one, where $(u_n)_+$ denotes the positive part of the normal displacement and C_n and m_n are two prescribed regularization parameters:

$$-R_n = C_n (u_n)_+^{m_n} \tag{8}$$

  This is very comfortable for the analysis and the resolution but let us look at the mechanical behavior. It has to be noted that we always have penetration into the obstacle with two consequences: one is a bad determination of the contact forces (which now depend on the compliance parameters) and the other one is a penetration into the obstacle that could be unrealistic.

  The term compliance is used when dealing with the mechanical model while the term penalization is used when solvers are involved.

Sometimes, in order to give a mechanical interpretation of this regularization, it is said that the penalization of the unilateral contact (or the compliance laws) is associated with the squeeze of the asperity. Actually, when the squeeze of an asperity is computed (in large plastic deformations, see Raous-Sage (94)), it turns out that the coefficients C_n and m_n are huge and do not correspond necessarily to the suitable values usually chosen to make computations easy to do. Therefore, it could be recommended either not to use regularization for the non-penetration conditions or - especially when a standard computational code is used - to check the penetration in order to control if its magnitude is admissible for the problem under consideration.

- Regularization of the Coulomb friction. In Fig 15, various functions used to regularize the Coulomb friction law are presented. Again, the Coulomb multivalued mapping is replaced by functions (which are non-linear) and this will make the problem a lot simpler.

  But it should be stressed that the new mechanical problem is totally different from the initial one: sliding always occurs, except if the applied tangential force is equal to zero! This means that with a very small force, you can get the refrigerator sliding across the kitchen if you wait for a sufficiently long period of time! Obviously, this is wrong!

Therefore, special attention has to be paid to the choice of the regularization parameters relative to the time scale. Very often, when the values of these parameters are chosen so as to give a good approximation of the Coulomb law, the regularized problem remains a stiff problem, ill-conditioned from a numerical point of view. When the values of these parameters are chosen for computational convenience, the quality of the solution has to be checked with respect to the Coulomb conditions, to be sure that this quality is sufficient for the problem.

Otherwise, the choice of the regularization parameters has a great influence on the tangential forces. Thus, let us emphasize that such regularization is not suitable for a subtle analysis, such as the study of instabilities or squeal, among others, because the results of the analysis will depend on the values of the regularization parameters (and even stability cannot be achieved if the regularization is strong).

Therefore, in conclusion - and this is very general in the art of modeling - regularization makes the mathematics, the formulations and the computations simpler but it transforms the problem in another one and we must be very careful when using it (choice of the parameters, verifications a posteriori on the solutions, etc.). Regularization is not always convenient.

Of course, regularization will be licit and used for problems where accurate determination of the contact condition is not needed.

2.2 Panorama of interface models

Beyond these basic laws which seem elementary but contain the essential characteristics for describing contact behaviors, many other laws have been developed and new ones are still under research. We will discuss a brief overview of these laws, without going into the details which can be found in the literature (references are given in (103)). This paragraph intends to show the variety of interface models.

Other friction laws.
In Fig. 16, configuration spaces for variants of the Coulomb law are given. These laws express that the friction no longer depends on the normal force when the latter becomes large. It is a kind of saturation of the friction threshold, which is often considered in metal forming. The Coulomb cone then becomes either a truncated cone (Coulomb-Orowan, Shaw) or a cylinder (Tresca).

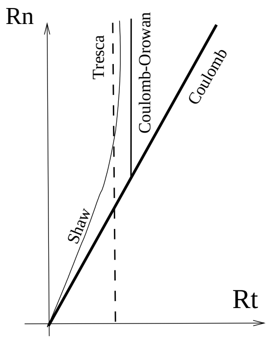


Figure 16. Friction cone for variants of the Coulomb law

Adhesion.
We now consider cases where the interface resists (with a resistive force) the separation of the solid from the obstacle. Therefore, the contact normal force can be positive. When the normal force increases, the adhesion progressively decreases as far as we finally turn back to the usual unilateral problem including friction. The intensity of the adhesion can be considered as a damage variable. These laws are used to describe interfaces as well as ductile cracks (Cohesive Zone Models CZM). More details can be found in the references given in Raous (103) and Sauer (109).

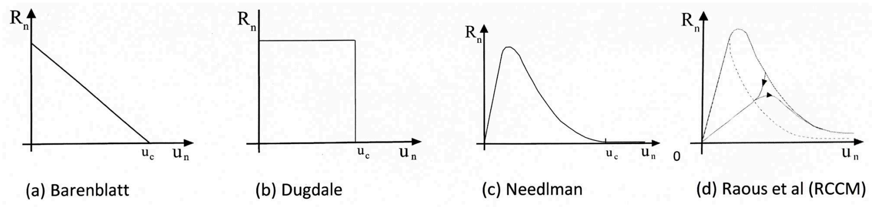


Figure 17. Adhesion laws for normal resistance

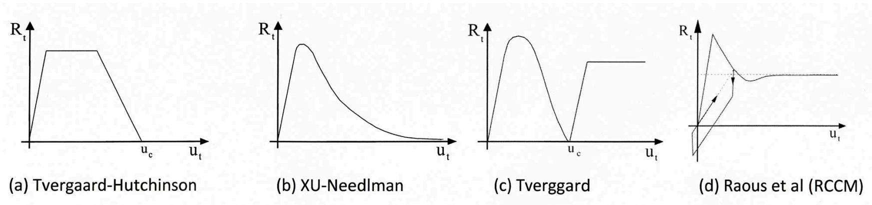


Figure 18. Tangential adhesion laws

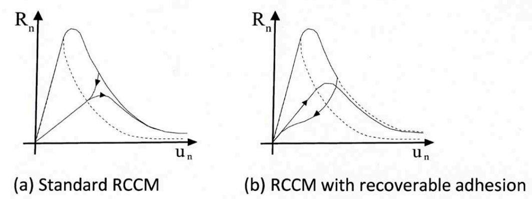


Figure 19. Recoverable adhesion

- Normal adhesion (resistance to traction). For the compressive force, we still have the strict unilateral conditions (Signorini problem). This is not regularization. In Fig. 17, some models are given.
- Tangential adhesion. These models are still based on the notion of intensity of adhesion i.e., of interface damage. When the adhesion totally collapses, one goes back to the usual Coulomb law. Some of these models are not coupled with friction. See Fig. 18.
- Recoverable adhesion. This is the case where, after total separation (and so total collapse of the adhesion), some adhesion is recovered when the solid is put again in contact with the obstacle (see Fig. 19). This could be a Van Der Walls force or some healing phenomenon. A model was proposed in Raous-Schryve-Cocou (100) (30). An example would be the case of an adhesive tape which can be used several times but which is a bit less efficient each time.

Wear, abrasion, grinding, polishing.
A number of laws regarding these phenomena can be found in the literature. The best known is the Archards law (7) for wear and abrasion modeling: the volume of the removed debris due to wear is proportional to the work done by friction forces.

Friction laws using a variable friction coefficient.
A dependence of the friction coefficient on the velocity, on the pressure, or other quantities is sometimes introduced. A classical law is the Stribecks law used for metal forming with a dependence of the friction coefficient on the velocity V_t, the normal pressure p and the viscosity of the lubricant b (see Fig. 20).

The dependence of the friction coefficient on the sliding velocity is also a classical feature (see Fig. 21).

However, great care is needed when such dependence (especially velocity dependence) is used in contact modeling. It is obviously legitimate when the velocity dependence can be clearly measured in some experiments. But it should be noted that dependence on the velocity is often introduced in the models too soon. For example, for a long time the reduction of the friction coefficient when velocity increases (or just considering static and dynamic friction coefficients) has been presented as a necessary condition for stick-slip to occur, which is not true: it is not a necessary condition. This will be shown in the example of Section 3.11.

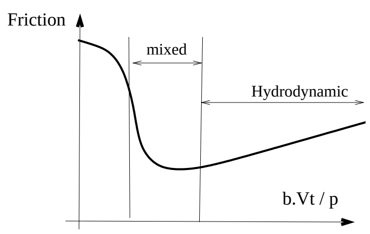


Figure 20. Stribeck law for metal forming

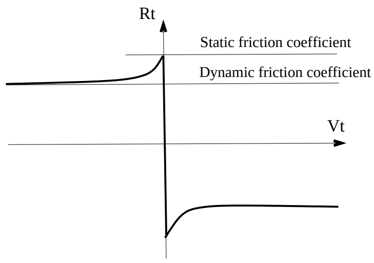


Figure 21. Variable friction coefficient

Interface models for faults in geophysics.
Earthquake rupture is classically modeled as a friction-dominated process, where friction arises from the roughness of the contact between the two sides of a fault. Earthquake initiation is strongly assumed to be triggered by friction instabilities. A lot of work has been done to characterize the conditions of instability and the very beginning of earthquakes (nucleation

duration and length). Choosing a realistic friction law for this interface is a key point for modeling nucleation of an earthquake and generation of the accompanying waves. Among many possible choices for the friction laws, it can be noted:

- slip-weakening friction (49) (120),
- rate-and-state friction (107) (17) (14).

The high number of parameters used in some of these laws is the sign of the complexity of the phenomenon to be modeled. In general, laws with many constitutive parameters should not be recommended. Despite the fact that giving convenient values (identification step) to these parameters will not be easy, choosing a model with many parameters could sometimes indicate that the model does not restrict to the dominant phenomena and to the main characteristics. That relates to the previous fundamental remark to make it as simple as possible and as complex as necessary. Research is ongoing on the use of the RCCM model (adhesion and friction coupling; 4 parameters) to describe the behavior of fault interfaces in geophysics (102) (121).

Conclusion.
In this section we have given an overview on the most usual interface laws in order to show the complexity of frictional contact modeling. We have pointed out some of the main issues regarding the art of modeling.

- A key point in contact analysis is the non-smooth character of the behaviors (multivalued mapping). Using regularization to eliminate this non-smooth character of the behaviors is not recommended.
- Make it as simple as possible and as complicated as necessary. That means keeping the number of variables and of parameters accounting for the dominant phenomena as small as possible (for example, it is better not to use a variable friction coefficient when it is not necessary).
- Constitutive assumptions are part of the model. It is important to keep in mind that a model is built for a certain purpose (no universality) and may provide an adequate description of a physical system under certain conditions (magnitude of the forces, velocities, etc .)

2.3 Choice of the scale in contact mechanics: various scales and multiscale analysis

Friction and adhesion are phenomena that can occur at very different scales, from the nanometer (thin films) to the kilometer (geophysics) and, of course, their physical and mechanical natures can be very different. This means that the physical ingredients to put in the model could be very diverse depending on the scale of the phenomenon.

For a given phenomenon, another point is the interactions between the global and the local behavior, for example between the scale of the rocks and that of the fault in geophysics, between the scale of the asperities and that of the plate in metal forming, between the scale of the molecule and that of a thin protective film layer.

For most models, the law is set directly at the global scale. Today, new approaches combine local and global analysis in order to put more physics into the global model. This is made possible by the advances made in tribology measurements and by the extraordinary increase in computer power allowing millions of degrees of freedom to be treated for numerical simulations.

Experimental knowledge at the local scale: micro tribology and asperity measurement.
Tribology is the science dedicated to the study and the experimental investigations of friction, wear, lubrication. Experiments and accurate measurements at the local scales (millimeter, micrometer and nanometer) have permitted a better understanding of the effects of normal pressure, speed, temperature and other environmental conditions on the tribological behaviors, and also a better understanding of breakout and movement of debris in an interface. Very interesting movies on this interface life and a theory of the third body in the interface have been proposed by Yves Berthier (13) at the LaMCoS in Lyon. This brings a strong contribution to the understanding of the global phenomenon, but the way to use and to put all these experimental measurements in the model is not easy. Measuring surface asperities has been done for quite a long time to characterize the plate surface in metal forming and stamping. They are described by their statistical characteristics. The hope would be to deduce the value of the friction coefficient from these statistical data. Introducing the local data into a global model is a big challenge.

Analytical approach.
First totally empirical, the Coulomb law got some justifications either from physical considerations by F.P. Bowden, D. Tabor (118) or local analysis by J.F. Archard (8) (9) or J.A. Greenwood(47). The Archards model was a first multiscale model. Archard approached the curve contact surface by a series of spheres and each sphere by another series of spheres of smaller radius (see Fig. 22) and so on. Then he used the analytical solution of the Hertz problem on each small sphere and got a global interface law. The Hertz solution is the analytical solution of the contact between a rigid sphere and

an elastic half plane (K.L. Johnson (55)).

Another example is the Greenwood-Williamson approach (47), based on a Gaussian asperity height distribution (see also B.N.J. Persson (85)).

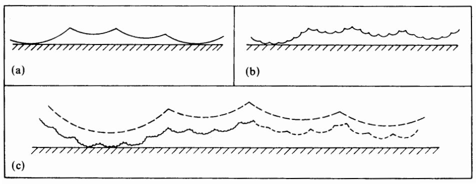


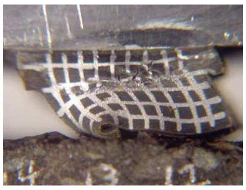
Figure 22. Archard modelization on the basis of sphere discretization and coulomb model

Mathematical approach.
Considering a third body with its own behavior, contact laws can be obtained by evaluating the limit of the solution when the thickness goes to zero. This has been conducted with simple (either linear or non-linear) behavior laws for the third body and has permitted to justify various contact laws. These mathematical tasks are complex because they deal with the notion of limit in a non-smooth context and they depend on the complexity of the chosen behavior law (Licht-Lichaille (67), Bouchitte et al (15), Dumont et al (39), Rizzoni et al (104), Serpilli (111)).

Numerical approaches - Multiscale simulations.
Over the years, some numerical methods have been developed for determining accurately the solutions in a localized zone in a structure where peculiar phenomena have to be taken into account and where refinement of the meshes is locally needed. Then iterative methods combining computations with several mesh levels have been developed: substructuration, FAQ methods, $(FEM)^2$, etc.). These methods avoid using a global refined mesh and so reduce the number of necessary DOF for determining an accurate solution to a problem including localized effects. The huge increase in computer power made computations with an extremely large possible number of DOF and also made it possible to mesh different zones with an extremely large refinement factor. These methods are of great interest for contact problems and have been used and developed in recent years. Let us cite the work of the research team led by Peter Wriggers at the ICM in Hannover, and that led by Jean-Franois Molinari at the EPFL in Lausanne.

To study the contact between a tire and the road and taking into account the asperities, Peter Wriggers and co-authors used multilevel approaches combined with a homogenization technique at the intermediate level. The problem is separated into macro and micro-scale problems. Thus local effects can be taken into account while keeping the computational time in a reasonable range, because the intermediate homogenization drastically reduces the computational time (80%). Details, applications and many references can be found in Temizer-Wriggers (119), Wriggers-Reinelt (130), De Laurenzis-Wriggers (34), Drosopouilos et al (42), Wagner et al (124), etc.

This multilevel methods make it possible to take into account local interactions, as the ones between the tire and road asperities, or the interaction with particles when a third body is modeled in the contact (Fig. 23).



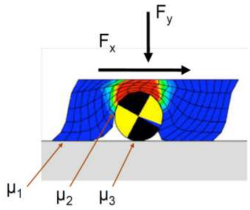


Figure 23. : Applications of multilevel approaches for contact problems (Courtesy of P. Wriggers)

Jean-François Molinari, Guillaume Anciaux and co-workers at the EPFL have developed multi-level models from atomistic to macroscopic levels. It can be observed that many surfaces tend to be nearly self-affine fractals (a self-affine fractal surface maintains its statistical properties when magnified).

In many numerical studies, from the atomic scale up to the geologic scale, a spectral generator of surfaces allows to employ representative surfaces and ultimately infer surface deformation mechanisms by connecting the contact properties with the statistical/fractal characteristics of the original surface (Yastrebov et al. (131), see Fig. 24).

The importance of the choice of the modeling scale was stressed in Section 1.2. Here the scale plays a paramount role. By using such profiles at the molecular level, the impact of the surface topology on all sorts of permanent deformations can be demonstrated. For instance, roughness flattening at the nano-scale leads to decreasing wear rates and to reduced friction coefficients (Spijker et al. (115) (116), see Fig. 25). In order to limit the cost of these simulations, routinely containing millions of atoms, Molecular Dynamics (MD) coupled to Continuum Schemes are employed (Anciaux et al (3), see Fig. 25).

With this multi-level model, two features have been outlined: the importance of the heat fluxes (see Ramisetti et al. (89) and the role of plastic deformations which takes the form of a collective motion of defects through crystalline materials (dislocations). In Fig. 26, this latter effect is well illustrated with a scratching indenter which deforms a bulk substrate and where some dislocations get trapped (Junge-Molinari (56)).

Recent advances in a three-way coupling between molecular dynamics, finite elements and discrete dislocations permit a powerful multi-level modeling and make it possible to simulate with greater accuracy sliding contact with consistent surface topologies which would reveal mechanisms often impossible to observe through experiments (see Cho et al. (19)).

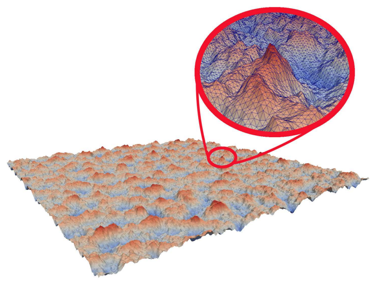


Figure 24. Discretization of surface asperities (courtesy of G. Anciaux)

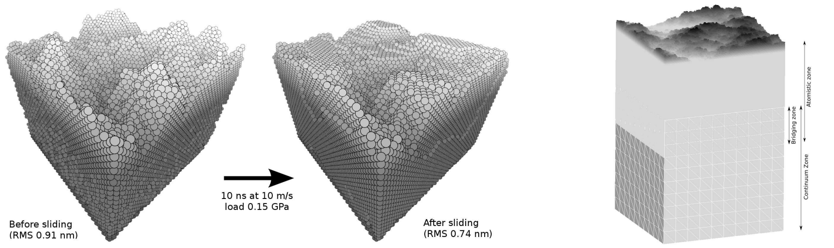


Figure 25. Flattening of nano-scale asperities during sliding and coupled MD-FE model for rough surface (2) (courtesy of G. Anciaux)

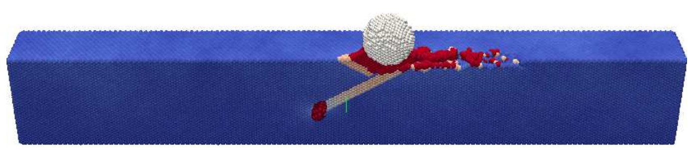


Figure 26. Simulation of a scratching indenter. (courtesy of G. Anciaux)

Conclusion about the choice of the scale in the art of modeling. In modeling and especially in contact modeling, great progress has been made on the experimental measurements, on the mathematical analysis and a lot on the computational capacities. Very interesting new opportunities have been opened towards a better understanding and a better modeling of contact behaviors by using local/global approaches:

- getting a better understanding and quantification of what friction or wear is, and what local behaviors are involved;
- modeling local effects of which some of them are
  - strong interaction between the solid and the asperities (same order of magnitude of the local deformations and the asperity size),
  - in tire-to-road contact, thin water film evolution and water flowing around the tire treads when it rains (various regimes), ice on the road, melting ice and cold temperature (coupling with thermal effects)
  - in metal forming, oil flows on Lasertex plates due to residual lubrication;
- connecting the global friction (friction coefficient or new global friction laws) and local surface characteristics (geometries, asperities, coupling, etc.).

2.4 Construction of a model coupling adhesion, friction and unilateral contact: thermomechanical and energetic basis.

In what follows, we want to illustrate with two examples the assertion made previously that models are based on concepts. The first example presents the construction of the RCCM model coupling unilateral contact, friction and adhesion. The second example is the generalization of this kind of models and the construction of a unified model including most of the models developed on this subject.

Models are based on general laws and conservation principles, complemented by constitutive relations which characterize the medium and should comply with some basic principles (causality, etc.). The RCCM (Raous-Cangémi-Cocou-Monerie) model is a model coupling adhesion, unilateral contact and friction. It is based on the concept of intensity of adhesion introduced by M. Frémond (45). Details on this model can be found in Raous (95), Raous et al. (96), Raous-Monnerie (99) and Cocou et al (25). We will stress the thermomechanical basis of the model and the difficulties due to the non-smooth character of the law. The application of the model to an industrial problem is presented: how to reduce crack propagation in a composite material by controlling the characteristics of the fiber/matrix interface.

The RCCM model: thermomechanical basis.
The problem is written here for the contact between two deformable solids and u_n denotes the gap and u_t the relative tangential displacement. Because of the convention chosen here, the unilateral conditions given by (9) are inverse to thoses given by (1) in Section 2.1.

The variables in the interface are:

- u_n, u_t the normal and tangential displacements,
- R_n, R_t the normal and tangential forces ($R_t^r = C_t\, u_t \beta^2$ is the reversible part of R_t),
- β the adhesion intensity (interface damage).

Unilateral conditions with adhesion

$$-R_n + C_n\, u_n\beta^2 \geq 0\ , \quad u_n \geq 0\ , \quad \left(-R_n + C_n\, u_n\beta^2\right)\, u_n = 0 \tag{9}$$

Coulomb friction with adhesion

$$\|R_t - R_t^r\| \leq \mu(1-\beta)\left|R_n - C_n\, u_n\beta^2\right| \quad \text{with}$$

$$\begin{array}{l} \|R_t - R_t^r\| < \mu(1-\beta)\left|R_n - C_n\, u_n\beta^2\right| \;\Rightarrow\; \dot{u}_t = 0 \\ \|R_t - R_t^r\| = \mu(1-\beta)\left|R_n - C_n\, u_n\beta^2\right| \;\Rightarrow\; \exists\lambda \geq 0\ ,\ \dot{u}_t = \lambda(R_t - R_t^r) \end{array} \tag{10}$$

Evolution of adhesion intensity

$$\dot{\beta} = -(1/b)\left(w\, - (C_n\, u_n^2 + C_t\, \|u_t\|^2)\beta\right)^- . \tag{11}$$

The parameters of the model are:

- μ the friction coefficient,
- C_n, C_t the initial stiffnesses of the interface,
- ω the adhesion energy (the Dupré energy) ,
- b the viscosity of the interface.

The graphs presenting the interface behavior of the RCCM model are given in Fig. 28 and 30.

The energies involved in the loading/unloading process are presented in Fig. 31. As was said before, the analysis of the model in terms of energy is fundamental.

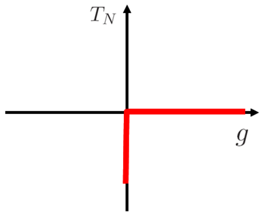


Figure 27. Signorini graph.

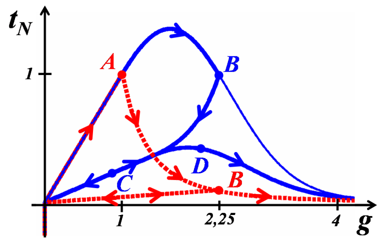


Figure 28. Normal behavior of RCCM model.

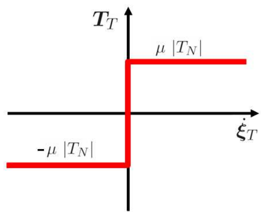


Figure 29. Coulomb graph.

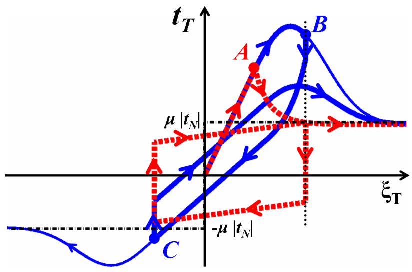


Figure 30. Tangentiel behavior of RCCM model.

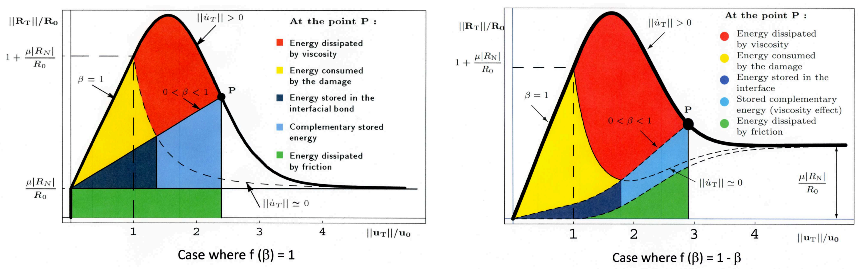


Figure 31. The energies involved.

The thermomechanical basis are the following:

- a hypothesis of material boundary is set for the contact zone: that means that we can associate to this part of the boundary a surfacic energy E and a specific entropy S and so the free energy of Helmoltz $\psi = E - ST$,
- the variables are u_n, u_t, β,
- the associated thermodynamical forces are R_n, R_t, G_β .

The next steps will be the following ones.

- Making an appropriate choice of the free energy ψ. It can be noted that it is non convex relative to (u_n, β) and non-differentiable. Then we apply the state laws in the sense of partial sub-differential and differential inclusions and we obtain the reversible part of the behavior.
- Making an appropriate choice for the potential of dissipation ϕ compatible with the Clausius Duhem inequality. It can be noted that it is non differentiable. Thus the complementary laws are written in terms of differentiable inclusions. We obtain the non-reversible part of the behavior (dissipation). The main steps are given below and details can be found in Raous (95) and Raous et al (96). The main objective is not to go into all the details in this course on the art of modeling but to emphasize the theoretical basis of the model both on the mechanical part (choice of the energies) and on the mathematical formulation (sub-differentials because of the non-convexity and differential inclusions because of the non-differentiability).

Reversible part of the behavior: choice of the free energy Ψ.

$$\Psi(u_n, u_t, \beta) = \frac{C_n}{2}\, u_n^2 \beta^2 + \frac{C_t}{2}\, \|u_t\|^2 \beta^2 - w\, h(\beta) + I_{\widetilde{K}}(u_n) + I_P(\beta) \tag{12}$$

where $\widetilde{K} = \{v \ / \ v \geq 0\}$ and $P = \{\gamma \ / \ 0 \leq \gamma \leq 1\}$. Introducting of the indicator functions $I_{\widetilde{K}}$ and I_P imposes the unilateral condition $u_n \geq 0$ and the condition $\beta \in [0, 1]$. To write the state laws, the two difficulties (lack of convexity and lack of differentiability) are overcome by using local or partial subdifferentiation. The state laws can then be written as follows :

$$\begin{aligned} R_n^r &\in \partial_{u_n} \Psi(u_n, u_t, \beta) && (13) \\ R_t^r &\in \partial_{u_t} \Psi(u_n, u_t, \beta) && (14) \\ -G_\beta &\in \partial_\beta \Psi(u_n, u_t, \beta) && (15) \end{aligned}$$

where ∂_u and ∂_β denote the subdifferential with respect to the variables u and β respectively. G_β is the thermodynamic forces associated the adhesion intensity β. The states laws give the reversible parts of the RCCM model.

Irreversible part of the behavior: choice of the dissipation potential Φ.
This potential agrees with the Clausius Duhem inequality.

$$\Phi\left(\dot{u}_t, \dot{\beta}; \chi_n\right) = \mu \left|R_n - C_n\, u_n \beta^2\right| \, \|\dot{u}_t\| + \frac{b}{p+1} \left|\dot{\beta}\right|^{p+1} + I_{C^-}(\dot{\beta}) \tag{16}$$

with $C^- = \{\gamma \in W/\gamma \leq 0\}$ and $p \leq 1$. The complementary laws are then written :

$$R_n^{ir} = 0 \quad (17)$$

$$R_t^{ir} \in \partial_{\dot{u}_t}\Phi(\dot{u}_t, \dot{\beta}; \chi_n) \quad (18)$$

$$G_\beta \in \partial_{\dot{\beta}}\Phi(\dot{u}_t, \dot{\beta}; \chi_n) \quad (19)$$

And we obtain the non reversible parts of the RCCM model (controling friction and adhesion).

Therefore we obtain all the relationships characterizing the interface model given at the beginning of this section.

The model has been constructed in two steps.

- Step 1: appropriate choices of the free energy and of the potential of dissipation compatible with the Clausius Duhem inequality.
- Step 2: the behavior laws are obtained by application of the state laws to the free energy and by application of the complementary laws to the potential of dissipation.

Application of the RCCM model to the fiber/matrix interface to reduce crack propagation in a SiC/SiC composite.

The industrial problem was to improve the resistance of a SiC/SiC composite to crack propagation by controlling the properties of the fiber/matrix interface. The objective was to build a model of the fiber/matrix interface behavior in order to check the influence of its characteristics on the crack propagation and to be able to give recommendations for the elaboration of the composite (enzyme, temperature, etc.) in order to improve its resistance. A model coupling unilateral contact, adhesion, viscosity and friction was elaborated. It was presented in the previous section.

Now in this section we will stress three steps of the modeling process: identification of the constitutive parameters, validation of the model and prediction. More details will be given in Sections 4.2, 4.3 and 4.5.

Identifying the constitutive parameters.

Identification of the four constitutive parameters was conducted in a fiber micro-indentation experiment carried out at ONERA (see Fig. 32). On Fig. 33 the good agreement between the experimental measures and the model results for these identified values can be noted.

Validating.

The model was validated by conducting other micro-indentation experiments operating on larger fibers, with other fiber volume fractions (sin-

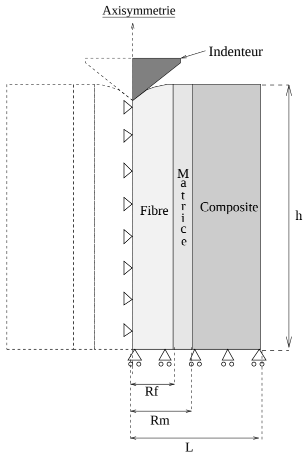


Figure 32. Model of the experiment.

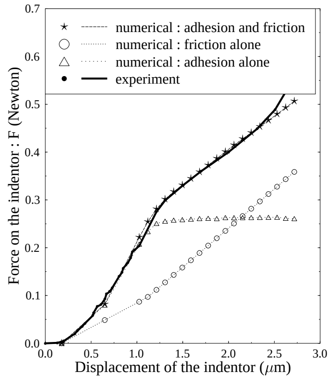


Figure 33. Identification of the constitutive parameters

gle fiber surrounded only with matrix) and with different thermal residual stresses. A good agreement was obtained.

Predicting.

The model was then used as a tool to optimize the interface characteristics in order to increase the resistance of the composite to crack propagation. The objective was to make recommendations on the suitable interface properties to improve the crack resistance of the material. When a crack propagates in a composite in a plane orthogonal to the fiber direction, a competition occurs between matrix crack bridging, matrix crack trapping by a row of fibers and fiber breaking. A 3D discretization of a single fiber surrounded with matrix was done. A prescribed displacement was applied on the two ends of the sample. A ductile model was used to describe the propagation of the crack. The global behavior was then evaluated with various values of the model parameters. Very different behaviors were observed and are presented in Fig. 34 and Fig. 35. For symmetry reasons, only half of the sample is represented, the crack propagates in the horizontal plane, the dark zones correspond to the total adhesion breakdown. In Fig. 34, the parameters put in the model correspond to a strong interface. We can observe that crack propagates both in the matrix and in the fiber and there is nearly no rupture along the fiber/matrix interface. Total rupture is obtained when the prescribed displacement reaches $0.12mm$. In Fig. 35, the parameters correspond to a soft interface. In this case, the crack propagates in the matrix, goes around the fiber but does not break the fiber and the interface

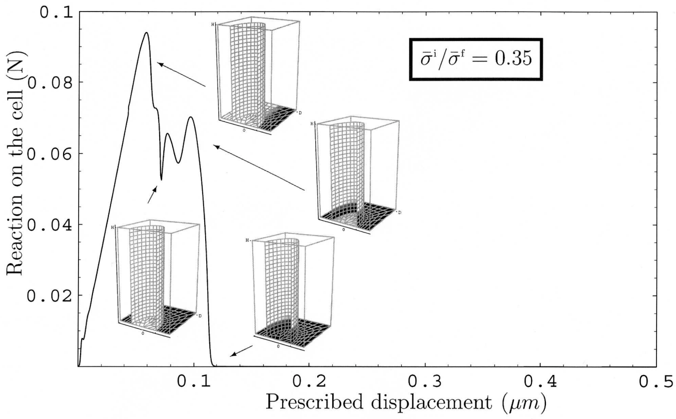


Figure 34. Crack progression (in black) and interface debonding in the case of strong interface

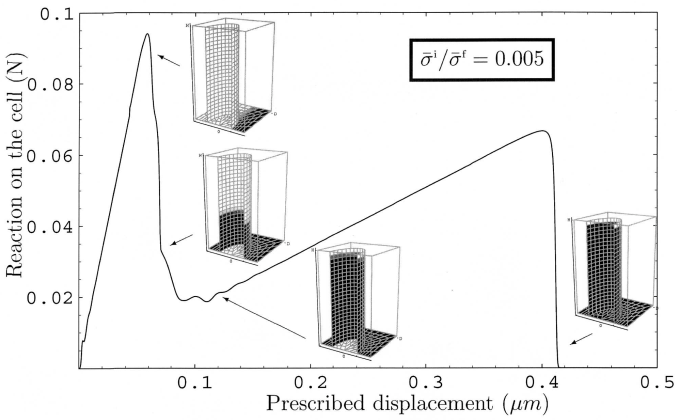


Figure 35. Crack progression (in black) and interface debonding in the case of soft interface

between the fiber and the matrix progressively collapses (energy dissipation). This time the total rupture occurs when the prescribed displacement reaches $0.4mm$. Thus the composite is three times more resistant.

This shows that a soft fiber/matrix interface will improve the composite resistance to crack propagation. Precise values can be given. Details can be found in Raous-Monerie (99).

2.5 A unified model for adhesion: inductive-deductive modeling process.

This work was published in Del Piero-Raous (35) (103) . Only the main features are given here. In that work, we showed how a set of various models (Cohesive Zone Model - CZM) can be gathered in a general unified model capturing the main features of the phenomena and based on fundamental concepts (thermomechanics, state space, choice for the energies and the dissipations).

To refer to the general comments given in Section 1, this work presents an up and down process:

- first an inductive process (giving sense to a corpus of data) gives the unified model
- then various models can be obtained as particular cases of the unified model , which is a deductive process (from general to particular).

The purpose of this work is to model a complicated interface response (unilateral contact, friction, adhesion, viscosity, etc.) with the smallest number of variables. This is conducted:

- by considering general laws, typically those of conservation of energy and dissipation of energy, that is, the mechanical version of the first two laws of thermodynamics,
- and choosing
  - a set of state variables, that is, an array of independent variables which fully determine the response to all possible deformation processes,
  - a set of an elastic potential and dissipation potentials, which are functions of state in terms of which the general laws take specific forms,
  - a set of constitutive assumptions.

The unified adhesion model is built as follows. Be given :

- a set of constitutive assumptions
  - a given loading curve (which could be either an experimental one or a prescribed behavior)
  - elastic behavior (with damage)
  - unilateral conditions
- a set of an elastic potential and dissipation potentials related to the effects that one likes to put in the model, among them but not only:
  - damage dissipation
  - viscous dissipation
  - friction dissipation

And by using the power equation and its derivatives when necessary, we determined the evolution of the damage variable.

Then, the usual adhesion models (and new ones) can be built from this general formulation by choosing conveniently the loading curve and the elastic and dissipation potentials. This was done in Del Piero-Raous (35) for the RCCM model.

3 Formulations, mathematical aspects, solvers

In Section 1, we discussed the choices of formulations and of the associated mathematical analyses as being important features for modeling. In this section, we want to stress the difficulties encountered in contact mechanics due to the non-smooth character of the laws. There are different kinds of difficulties:

- contact laws are multivalued (non-differentiable) mappings and not functions,
- the friction law is non-associated (no normality rule),
- the variables are defined on the boundary (trace space),
- shocks may occur, time discontinuities may arise and the solutions will be not differentiable.

Thus, formulations are not mathematically elementary. We need to use variational inequalities, differential measures, differential inclusions, etc. Also, solvers have to deal with this non-smooth character of the contact laws and more difficult problems arise.

However - and this may be the most important point to be emphasized in this section - this non-smooth character is essential in the model to deal with the complex phenomena which can be observed with friction. Modeling some phenomena observed in contact mechanics requires that these theoretical specificities associated with this non-smooth character were considered and dealt with correctly. This means that for an accurate description of the contact behavior, regularization should be avoided as often as possible. This is illustrated by the example of the squeal of waist seal sliding on a window glass presented Section 3.11. However, there are of course cases where a simplified model of contact behavior may be sufficient (for example modeling crash tests). We present first the static problem which is a displacement formulation. It has no physical meaning, as friction should be expressed in terms of velocity, but it will be an intermediate problem, very useful for solving the problem expressed on the velocities and moreover the main difficulties can be identified on this formulation.

3.1 Formulations of the "static" problem

As presented in Fig. 36, the solid occupies the domain Ω and is submitted to boundary conditions on Γ_D and to volume and surface loadings in Ω and on Γ_F. On the part Γ_C of the boundary Γ we have unilateral and friction conditions. The solid is here supposed to be elastic.

Problem P_{stat}: Let Φ_1, Φ_2 be the given forces, find the displacement field u, the stresses σ, the strains ϵ and the contact force R such that :

$$
\left\{
\begin{array}{l}
\text{Elastic behavior and the equilibrium} \\
\left.\begin{array}{lcl}
\varepsilon & = & \mathrm{grad}_s\, u \\
\sigma & = & K\varepsilon \\
div\sigma & = & -\phi_1
\end{array}\right\} \text{ on } \Omega \\
\text{Boundary conditions} \\
\begin{array}{ll}
u \ = 0 & \text{on } \Gamma_D \\
\sigma.n \ = \phi_2 & \text{on } \Gamma_F
\end{array} \\
\text{Unilateral contact with friction} \\
\left.\begin{array}{l}
\sigma.n \ = R \\
u_n \ \leq 0 \\
R_n \ \leq 0 \\
u_n R_n \ = 0 \\
\|R_t\| \leq \mu \mid R_n \mid \ \text{with} \\
\text{if } \|R_t\| < \mu \mid R_n \mid \ \text{then } u_t = 0 \\
\text{if } \|R_t\| = \mu \mid R_n \mid \ \text{then } \exists \lambda > 0 \text{ such that } u_t = -\lambda R_t
\end{array}\right\} \text{ on } \Gamma_C
\end{array}
\right.
$$

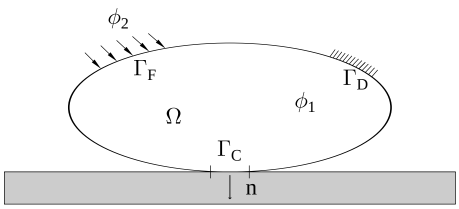


Figure 36. The solid

3.2 Variational formulation

For a classical elasticity problem without contact conditions, the variational formulation is written under the form of a variational equation. For frictional contact problems, we get an implicit variational inequality (see Duvaut-Lions (40), Raous (95)) or a quasi-variational inequality when the dual formulation is considered (see Panagiotopoulos (83)).

Let K be the convex of the admissible displacements (2D formulation) $K = \{v \in U / \ v_n \leq 0 \text{ on } \Gamma_C\}$ with $U = \left\{v \in \left[H^1(\Omega)\right]^2 / \ v = 0 \text{ on } \Gamma_D\right\}$.

Problem P_{var}: let $\phi_1,\ \phi_2$ be given as previously defined in *Problem* P_{stat}, find u ∈ K such that :

$$a(u, v-u) + J_1(u, v) - J_1(u, u) \geq L(v-u) \qquad \forall v \in K \tag{20}$$

with :

$$a(u, v) = \int_\Omega \sigma(u)\varepsilon(u)\ dx = \int_\Omega E_{ijkl}\varepsilon_{ij}(u)\varepsilon_{kl}(v)\ dx \qquad \forall u, v \in U \tag{21}$$

$$L(v) = \int_\Omega \phi_1 v\ dx + \int_{\Gamma_F} \phi_2 v\ ds \qquad \forall v \in U \tag{22}$$

$$J_1(v, w) = \int_{\Gamma_C} \mu |F_n(v)| \|w_t\|\ ds \tag{23}$$

where $a(.,.)$ is the classical bilinear form associated with the elasticity operator E and $L(.)$ is the linear form associated with the loadings (it represents the work of the loads in the virtual displacement v). The form $J_1(.,.)$ is associated with the friction (it represents the work of the tangential contact force; note that the normal force does not work because of the complementarity condition).

3.3 A minimization problem (fixed point method).

It is not possible to associate a minimization problem (minimum of the potential energy) with the variational problem *Problem* P_{var} as can be done in classical elasticity. This is due to the non-associated character of the Coulomb's law, since the sliding velocity does not satisfy the normality rule (see section 2.1). For both mathematical and numerical reasons, it will be helpful to set an equivalent form of the previous problem by using a fixed point on the sliding limit, associated with a Tresca problem. For the Tresca friction, the sliding limit does not depend on the normal force and the sliding velocity satisfies the normality rule, because the Coulomb cone has been replaced by the Tresca cylinder (see Fig. 16). Thus we obtain the following problem.

Problem P_{fp}: Find the fixed point of the application S :

$$S(g) = -\mu F_n(u_g) \tag{24}$$

with u_g solution of the following problem $P_{varTresca}$:

Problem $\boldsymbol{P_{varTresca}}$: For a given g, find $u_g \in$ K such that :

$$a(u_g, v - u_g) + j(v) - j(u_g) \geq L(v - u_g) \qquad \forall v \in K \tag{25}$$

with: $j(v) = \int_{\Gamma_C} g\|v_t\| \, ds$

The previous Tresca problem is then equivalent to the following minimization problem :

Problem $\boldsymbol{P_{mini}}$: For a given g, find $u_g \in K$ such that

$$J(u_g) \leq J(v) \qquad \forall\, v \in K \tag{26}$$

with $J(v) = \frac{1}{2}a(v, v) + j(v) - L(v)$

The problem is now set as a minimization problem under constraints of a non-differentiable functional which needs be solved for each value of the sliding threshold obtained at every step of the fixed point application.

3.4 Alternative formulations

Many other formulations can be given for the initial $Problem\ P_{stat}$ and each will need different kinds of solvers. It is now easy to understand why the choice of the formulation is one of the important choices in the art of modeling, as was said in Section 1.3. Without going into details, we can cite the following formulations.

Complementarity problem.
Another approach consists in writing the problem in the form of a complementarity problem (see Cottle et al (31)), introducing two new friction variables by separating the tangential displacement into left and right sliding parts (see Raous (95)). In the 2-dimensional case, it is then written, after FEM discretization and condensation in reducing the problem to the contact variables (partial inversion of the linear parts), as $Problem\ P_{compl}$.

Problem $\boldsymbol{P_{compl}}$: Find $F \in \Re^p,\ u \in \Re^p$ such that

$$\left.\begin{array}{l} Mu = F^* + R \\ R_i \leq 0,\ u_i \leq 0 \qquad i = 1 \ldots p \\ R_i u_i = 0 \qquad i = 1 \ldots p \end{array}\right\} \tag{27}$$

where M and F^* are respectively a non-symmetric matrix and a loading vector deduced from the FEM problem by condensation (and taking into account the change of variables associated to the choice of new variables), R and u are the contact forces and the contact displacements, and p is the number of contact degrees of freedom. This 2D formulation has been extended to 3D problems using a polygonalization of the Coulomb cone by Klarbring (61).

Penalization formulation for the frictionless contact.
The penalization principle consists in adding an extra force G_ϵ defined on the contact boundary to enforce the conditions $u_n \leq 0$; a penalization parameter ϵ is introduced and a nonlinear variational equation is obtained.

Problem P_{penal}: Find $u \in U$ such that $\forall\, v \in U$

$$a(u, v) = L(v) + G(v) \tag{28}$$

with $G(v) = \int_{\Gamma_C} G_\epsilon \; v \; ds$

Normal penalization can be considered as a numerical form of the compliance law previously introduced in Section 2.1.

Lagrange multipliers. In that case, the contact force is kept as a variable (Lagrange multiplier). It is a mixed formulation. We get a saddle point formulation which is a min/max problem.

Augmented Lagrangian. It is a combination of the penalty and the Lagrange multiplier formulations.

Details of all these formulations can be found in the books by Laursen and by Wriggers (63) (128) (129).

3.5 Formulations of the quasi-static problem

Coulomb friction law has to be expressed on velocities. It was shown in Cocou-Pratt-Raous (22) (23) (24) (27) and Shillor et al (110) that the problem is then written as the coupling of two variational inequalities (one of which is implicit). The problem can be written as follows.

Problem P_{qs}: For t belonging to $[0, T]$ and with presribed initial conditions, find $u(t) \in K$ such that :

$$\begin{aligned} & a(u(t), v - \dot{u}(t)) + J_1(u(t), v) - J_1(u(t), \dot{u}(t)) \geq L(v - \dot{u}(t)) \\ & \qquad\qquad + < R_n(u(t)), v_n - \dot{u}_n(t)) > \qquad \forall v \in V \\ & < R_n(u(t)), z_n - u_n(t)) > \;\; \geq 0 \qquad \forall z \in K \end{aligned} \tag{29}$$

3.6 Formulation of the dynamics problem.

The main contributions to this topic are from Jean-Jacques Moreau (75) (76) for finite dimensional problems (granular medium). In $Problem\ P_{stat}$, the equilibrium equation should be replaced by the equation of motion:

$$\rho\, \ddot{u}(x,t) = \mathrm{div}_x \sigma(u(x,t)) + \phi_1(x,t) \tag{30}$$

As was noted before, in the case of a contact problem, the occurrence of impacts and shocks has to be considered and thus the velocities are not continuous (and not differentiable). The acceleration cannot be defined in the usual sense, the notion of differentiable measures has to be used. For the sake of simplicity, we give directly the discrete formulation of the problem.

Problem $\boldsymbol{P_{dyn}}$: Find U such that
$\forall t \in [0,T]\ U(t) \in V_h$, $U(0) = U_0$, $\dot{U}(0) = V_0$ and :

$$M.d\dot{U}\ +\ K.U + C.\dot{U}\ =\ F\ +\ Rd\nu \tag{31}$$

and should be satisfied the Signorini and Coulomb conditions for the contact nodes (Section 2.1).
$d\dot{U}$ is a differential mesure representing the discretized acceleration and $d\nu$ is a nonnegative real mesure relative to which $d\dot{U}$ happens to possess a density function. The differential measure is a generalization of the notion of derivative which takes into account the jumps. The derivative $\dot{u} = du/dt$ is replaced by a differential measure du (Stieltjes measure). In the smooth case (u is continuous), we have $du = \dot{u}dt$ where dt is the Lebesgue measure, which is in fact the differential measure of the (real) function t. In the general case, for any compact sub-interval $[a,b]$ we have :

$$\int_{]a,b]} d\dot{U}\ =\ \dot{U}^+(b) - \dot{U}^+(a) \tag{32}$$

with right continuity : $\dot{U} = \dot{U}^+$
This formulation in terms of differential measure is the convenient formulation to be used for dealing with the jumps and the shocks which may occur in contact dynamics. Convenient numerical methods to solve this problem set under this sophisticated formulation will be given in Section 3.8.

3.7 Mathematical analysis

In this section, we want to stress that some problems could be very stiff, considering the mathematical properties of the operators and of the solutions.

The art of modeling is then to cope with some choices:

- either transforming the initial problem (regularization, etc.) in order to get a simpler model, easier to solve, but , as was underlined before, the regularized problem is a problem different from the initial one; we should be aware of that a feature because regularization is currently used in most computer codes,
- or addressing the real difficulties of the initial problem, which is not easy to do, but is the right way to get the correct solutions, even if this way is complex.

The mathematical analysis of the non-smooth problems set when modeling contact phenomena is very important. It is essential for a good understanding of the solutions that can be obtained and for overcoming the difficulties which arise both in the formulations and in the solvers. As was said in Section 1, a large scientific culture is necessary (either individually or collectively as a team). It is impossible to be specialist of everything but it is important to have some knowledge of these various topics, including the mathematical topics presented in this section.

Main difficulties and some alternatives.
The situation can be summarized as follows.

- The behavior laws are represented by multivalued mapping.
  Consequence: variational inequalities.
  Alternative: regularization, but we get a very different problem.
- The Coulomb law is non-associated.
  Consequence: no minimum principle.
  Alternative: Tresca problem (as intermediate problem) + fixed point on the sliding threshold.
- The contact force is a distribution belonging to $H^{-\frac{1}{2}}$ (this point will be addressed below).
  Consequence: compactness issues arising in the mathematical analysis.
  Alternative: new definition of the contact forces defined by a convolution with a smooth function with compact support (non-local friction).
- For the dynamic problem : impacts result in velocity discontinuities.
  Consequence: differential measure formulation.
  Alternative: no alternative in the general case; nevertheless for rigid body dynamics, restitution coefficients is often used.

Mathematical framework and difficulties.
The following scheme (in 3D) gives a synthetic representation of the mathematical framework for contact problems.This is very helpful to understand the reasons for the difficulties encountered in the mathematical analysis of contact problems.

Duality pairings

$$
\begin{tikzcd}
\text{Contact} & U_c = H^{\frac{1}{2}}(\Gamma_C)^3 & (R,w) & \Phi_c = H^{-\frac{1}{2}}(\Gamma_C)^3 \arrow[d] \\
\text{Displacement} & U = H^1(\Omega)^3 \arrow[u, "\begin{array}{c}\text{Trace}\\ \text{operator}\end{array}"'] & <\Phi, v> & \Phi_F = L^2(\Omega)^3 \mathrm{x} L^2(\Gamma_F)^3 \arrow[d, "\begin{array}{c}\text{Equilibrium equations}\\ \mathrm{div}\sigma = -\phi_1 \text{ in } \Omega\\ \sigma.n = \phi_2 \text{ on } \Gamma_F\end{array}"] \\
\text{Strains} & E = L^2(\Omega)^9 \arrow[u, "\begin{array}{c}\text{Deformation}\\ \text{operator}\\ \epsilon = \mathrm{Grad}_s u\end{array}"'] & << \sigma, \varepsilon >> & S = L^2(\Omega)^9 \ \text{Stress}
\end{tikzcd}
$$

$$<< \sigma, \varepsilon >>= \int_\Omega \sigma(u)\varepsilon(v)\, dx \qquad \forall v \in U \tag{33}$$

$$< \Phi, v >= \int_\Omega \phi_1 v\, dx + \int_{\Gamma_F} \phi_2 v\, ds \qquad \forall v \in U \tag{34}$$

$$(R, w) = \int_{\Gamma_C} \mu |R_n(u)| \|w_t\|\, ds \qquad \forall w \in U_c \tag{35}$$

This scheme has 3 levels. Levels 2 and 3 (at the bottom) present the classical scheme for an elasticity problem.

- At level 3, the stress space S and the strain space E are $L^2(\Omega)^9$; they are in duality according to the duality product $<< \sigma, e >>$ which represents the work of the stress σ in the strain ϵ,
- At level 2, the displacement space U is $H^1(\Omega)^3$ and the loading space ϕ_F should be the dual of $H^1(\Omega)^3$ for the duality product $< \phi, v >$ (which represents the work of the loadings (ϕ_1, ϕ_2) in the displacement u); however, as the loadings are given, we are free to choose them in a more regular space and we choose $L^2(\Omega)^3 \mathrm{x} L^2(\Gamma_F)^3$. This is a key point for the study of the existence and the uniqueness of the solution to the mathematical problem set for a small deformation elasticity problem.
- Level 1 is the level of the contact variables. The situation is similar but, as the contact forces are unknown, it is not possible to choose a more regular space for them. The contact displacement space U is $H^{\frac{1}{2}}(\Gamma_C)^3$ (trace of u on the boundary Γ_C) and the contact force is in $H^{-\frac{1}{2}}(\Gamma_C)^3$, which is the dual of U for the duality product (R_c, u_c). $H^{-\frac{1}{2}}(\Gamma_C)^3$ is a distribution space, which will make the mathematical analysis more complicated .

To complete this schema, we have indicated the mappings acting from one space to another. Let us note again that each duality product represents a work. This gives a good picture of the mathematical situation.

Introduction of non local friction.
In classical elasticity, existence and uniqueness theorems are based on coerciveness and continuity of the operators, and the proof is based on the Lax-Milgram theorem and Cauchy-Schwarz inequality (equivalence of norms).

In contact mechanics, the first difficulty which arises is that the contact force is a distribution belonging to $H^{-\frac{1}{2}}(\Gamma_C)$, as explained in the previous section. This implies some compactness difficulties, and a regularization R^{reg} using a convolution product is used to overcome this difficulty.

$$R^{reg} = R * \psi \tag{36}$$

where ψ is a very smooth function with compact support (see Duvaut (41) and Cocou (21). Thus, the space for the contact force is much smoother and this will be very helpful for the mathematical analysis:

- R is in $H^{-\frac{1}{2}}(\Gamma_C)$, it is a distribution,
- R^{reg} is in $L^2(\Gamma_C)$, it is a function.

However, from a mechanical point of view, it should be noted that the contact forces are now defined by using a notion of non-local forces.

Overview of the main results about existence and uniqueness of solutions.
With this overview of the main mathematical results on contact problems, we want to call attention to the consequences of the non-smooth character of the law. The mathematical difficulties which are pointed out have direct consequences on the art of modeling, both on the kind of phenomena that can be observed and on the computational difficulties that may be encountered.

- Static problem (no mechanical meaning but interesting intermediate problem). The implicit variational inequality (20) in *Problem* P_{var} needs to be solved.
  - Signorini problem (no friction) (112):
    existence and uniqueness of the solution - Fichera (44);
  - Signorini + Coulomb friction:
    existence if μ is small and no uniqueness - Necas-Jarusek-Haslinger (79), Jarusek (50), Eck-Jarusek (43);

- Signorini + Coulomb (non-local friction):
  existence and uniqueness if μ is small - Cocou (21), Duvaut (40), Demkowicz-Oden (37);
- Normal compliance + Coulomb friction:
  existence and uniqueness if μ is small - Klarbring-Mikelic-Shillor (59);

• Quasi-static problem. The two coupled variational inequalities of Problem P_{qs} (one of them is implicit) need be solved.

  - Signorini + Coulomb:
    existence if μ is small (both in L^∞ and as a multiplier in $H^{-\frac{1}{2}}$) and no uniqueness of the solutions - Andersson (5), Cocou-Rocca (26) (105) (106);
  - Signorini + Coulomb (non-local friction):
    existence if μ is small (condition only in L^∞) and no uniqueness - Cocou-Pratt-Raous (22) (23);
  - Compliance + Coulomb:
    existence if μ is small and no uniqueness (only a few works) - Andersson (4), Klarbring-Mikelic-Shillor (59);
  - Examples of non uniqueness for a discrete problem - Ballard (10).

• Dynamic problem. The *Problem* P_{dyn} is formulated in terms of differential measures. There are very few mathematical results in elasticity, more results in viscoelasticity.

  - Continuous problem
    * frictionless in elasticity:
      normal compliance: existence - Martins-Oden (70) (71)
      Signorini: a few results on specific geometries (axial symmetry) - Munoz-Rivera-Racke (78);
    * normal and tangential compliance in viscoelasticity:
      existence and uniqueness - Martins-Oden (70) (71), Kuttler (62);
    * Signorini problem + non local friction in viscoelasticity:
      existence - Cocou (28), Cocou-Scarella (29);
    * Signorini + Tresca friction in viscoelasticity:
      existence - Jarusek (51);
  - Discrete problem
    * existence and uniqueness for analytical loading in 1D
      · frictionless - Ballard (11);
      · with friction - Ballard-Basseville (12).

Concluding remarks on mathematical formulations.
Contact and friction relate to non-smooth mechanics: the interface laws are multivalued mappings, the formulations are written in terms of implicit variational inequalities or differential measures for dynamic problems, in most cases there is no-uniqueness - and sometimes nonexistence - of solutions, and some problems remain unsolved (no results).

At this point, it is essential to note that the occurrence of multiple solutions is not a fantasy of mathematicians: it can really occur. Klarbring (60) and other authors (Janovsky, Alart-Curnier, Mitsopoulos-Doudoumis) constructed very simple examples with a few masses and springs showing the existence of two solutions, the occurrence of which depends on the values of μ. Unfortunately, the mathematical conditions set on μ depend on mathematical constants such as the coerciveness or the continuity constants that cannot be easily evaluated in terms accessible to the designer.

What should be remembered is that multiple solutions may occur and that the bad conditioning of the problem increases when μ increases. This is also observed with the computational resolution. All this is directly connected with the mathematical results given above.

From a mechanical point of view, this bad conditioning for large values of μ will be observed during the studies on stability. In Section 3.11, this will be illustrated with the presentation of a study on the stability analysis of a mechanical system and the search for occurrence of unstable solutions to modeling squeal phenomena.

In closing this section, we want to underline the strong relationship between mechanics and mathematics, in both directions, that is, from mechanics to mathematics and vice versa. In mechanics, mathematical tools are often used while important developments in mathematics have been motivated by the mathematical study of mechanical problems and especially of contact mechanics problems; for example, variational inequalities, convex analysis, Γ-convergence, etc.. These advances are due to specialists in theoretical mechanics and mathematicians (mostly by a French-Italian school and some famous representatives from the USA and Greece). Without trying to be exhaustive, let us cite Jacques-Louis Lions, Georges Duvaut, Jean-Jacques Moreau, Panagiotis Panagiotopoulos, Tyrell Rockafellar, Enrico Magenes, Guido Stampacchia, Umberto Mosco, Bernard Nayroles, Pierre Suquet, Marius Cocou, Patrick Ballard, Yves Renard, Patrick Hild, Michelle Schatzman, Laetitia Paoli, etc.). Let us mention the reference book by R. Dautray and J.L. Lions (33).

Highly theoretical work is still in progress in contact dynamics, along with sophisticated mathematical developments.

3.8 A short overview of the solvers

The topic is broad and work addressing this topic is still in constant development. The idea is obviously for this course not to provide an exhaustive presentation of the solvers used in contact mechanics but to provide some indications and to stress the assumptions that they implicitly make. In fact, we want primarily to draw the attention of the engineer on the care that must be taken in choosing a solver or in choosing a computer code to solve a contact problem. It is essential to check what kind of methods are used, what numerical parameters are involved and to make some control tests on the numerical solution.

Details on numerical methods for contact problems can be found for example in the books by Wriggers (128), Laursen (63), Wriggers-Panagiotopoulos (Eds) (127), Wriggers-Laursen (129), Kikuchi-Oden (58). Let us first have a look at the methods that are currently used to solve quasi-static problems. Some of these methods are commonly used in commercial computer codes.

Penalty formulation.
As presented in Section 3.4, penalization is a regularization of the strict contact conditions. An extra force G_ϵ, defined on the contact boundary, is added to enforce the unilateral condition $u_n \leq 0$ and a condition controlling the friction. Two penalization parameters ϵ_n and ϵ_t (shortly noted by ϵ) are introduced. Normal penalization can be considered as a numerical form of the compliance law previously introduced. The functions G_ϵ are mostly nonlinear functions with a stiff dependence on the normal penetration into the obstacle (or sometimes on the vicinity of the obstacle in case of external penalization) or on the tangential sliding displacement. A nonlinear variational equation (28) is then obtained. After discretization, a nonlinear problem needs to be solved and Newton Raphson methods are usually used (see Alart-Curnier (1)).

$$Au = F + G_\epsilon(u) \tag{37}$$

Two remarks have to be made. First, to conveniently ensure the contact conditions the computing parameters ϵ_n and ϵ_t have to be chosen in such a way that the penalization functions are often stiff and thus the numerical problem remains hard to solve (bad behaviors of the algorithms). When penalization is soft, computations are easier but contact conditions are then often roughly fulfilled. Secondly, as noted in Section 2.1, penalization leads to an approximation model which is different from the initial model. Thus, two verifications should be done. The first one, relating to the model, is to make sure that non-smooth solutions are not of interest in the phenomenon studied (such as oscillations, instabilities, etc.) because they are usually lost when regularization is used. The second one, concerning the numerical solution, is to check that the values of the penetrations into the obstacle and of the micro sliding amplitudes fit the model objectives and the accuracy expected for the solution. The regularization drastically disrupts the determination of the contact forces.

In commercial computer codes, regularization parameter values are often proposed as default values, but specific choices of the function $G_\epsilon(u)$ (both the function shape and the parameters ϵ_c) are generally offered and it is recommended to use them. It will be observed that the choices of the penalization parameters improving the quality of the solutions often lead to large computational times. In any cases great care is recommended when regularization is used.

Lagrange multiplier formulation.
In that case, the contact force is kept as a variable (Lagrange multiplier). It is a mixed formulation. This method permits an accurate determination of the contact forces. We get a saddle point formulation which is solved by using the Uzawa algorithm (min/max optimization). It should be noted that additional variables (the contact forces) have to be introduced in that case (larger number of DOF).

Augmented Lagrangian formulation (widely used).
(see Simo-Laursen (114))
It is a combination of the penalty and the Lagrange multiplier formulations. An iterative process is used to reach the correct value for the multiplier (number of augmentations). The augmentation consists in adding a penalization term. The case with only one augmentation corresponds to the usual penalization. We get a variational equation coupled with the conditions of contact and friction (Kuhn Tucker conditions). A Newton Raphson method, which depends on the number of augmentations, is combined with a radial return process for the contact. For friction, stick prediction is done and correction is conducted for sliding when it is needed.

This method is very powerful and very often used in the computer codes developed in academic research laboratories.

We present now numerical methods solving the strict conditions for unilateral contact and friction. Details can be found in Raous (95), Raous et al (92), Chabrand et al (16), Lebon-Raous (64), Klarbring-Björkman (61).

Lemke method (complementarity problem).
The complementarity formulation of the contact problem complying with the strict contact conditions was given in section 3.4 (see Klarbring-Björk-

man (61), Raous (95) (90), Cottle et al. (31)).

Methods derived from mathematical optimization, such as the Lemke method or the interior point method (see (20) (61)) can be used. The Lemke algorithm is a mathematical programming method. It is a direct method based on pivoting techniques, similar to the Simplex method.

In elasticity or for linearized behaviors, a condensation of the problem can be conducted in order to write the problem only on the contact variables. This is done by a partial inversion of the total system that can be performed not by inverting the global matrix but by solving (only once at the beginning) a set of linear problems. Details are given in Raous (95), an extension to treat dynamic problems is given in Vola et al (122). Then the Lemke method is applied on the reduced system whose rank depends only on the number of contact nodes. It is however a full sized matrix which is non-symmetric (because of the friction). Using a direct method is very comfortable; the number of pivotings is less than the rank of the matrix (and in practice much more less). However, because of the full sized matrix, this process cannot be used for huge numbers of DOFs. In that case, when subdomain or multigrid methods are used, the Lemke method turns out to be still very efficient for coarse grids (see Lebon et al. (65)).

Fixed point on the sliding limit and minimization problem.
It has been shown that the variational inequality problem (strict contact conditions) can be set as a sequence of minimization problems combined with a fixed point method on the sliding limit. At each step, we solve a Tresca problem, i.e., a frictional problem where the sliding limit is given. The Trescas law being an associated law, a minimum principle can be associated. The iteration on the sliding limit converges quite fast (less than 10 iterates and often less than 5) depending on the size of the system and on the prescribed accuracy. Details can be found in Raous (95).

We have to solve $Problem\ P_{mini}$ which is a minimization problem under constraints (u belongs to the convex K) of a non-differentiable functional. Various minimization methods with projection can be used.

- Successive Over Relaxation and Projection (SORP). An optimal parameter of relaxation has to be determined using a trial procedure. The method is very robust but can be costly when extension to non-linear problems is considered.
- Gauss-Seidel with Aitken acceleration. No numerical parameter is needed.
- Pre-conditioned conjugate gradient with projection (see Raous-Barbarin (93)). This is a very powerful method but a regularization of the friction term has to be done in order to evaluate the gradient of the functional to determine the descent directions.

Multigrid methods. see Lebon-Raous-Rosu (65).
As noted before, we have developed multigrid methods for solving contact problems without regularization of the contact, i.e., using the strict Signorini conditions and the strict Coulomb law.

Multigrid methods operate at several levels of meshes (usually 2 to 5 levels) which are coarser than the initial mesh where the solution will be calculated. This is an iterative process using complete resolutions on the coarsest grid and a few smoothings conducted on the default of equilibrium on the intermediate grids. At each iteration, the Lemke method can be used on the coarsest grid to solve the small sized problem, and the projected Gauss Seidel method will be a very efficient smoother on the other finer grids.

In all the examples presented in this course, for both quasi-static and dynamic problems, no regularization of the contact conditions was used and the algorithms fitted the non-smooth character of the laws. This is fundamental for modeling specific phenomena such as squeal, which is due to the occurrence of unstable solutions for the theoretical problem.

Dynamics problem.
As presented before, because of the non-derivability of the solutions (shocks), the problem has to be formulated in terms of differentiable measures: equation (31).

A classical Newmark method which is based on a limited development of the solution needs their derivability and cannot be used. However, some improvements have sometimes been developed to adapt the Newmark method to the problem (usually some numerical damping). But, we should be aware that the use of numerical damping will kill not only numerical oscillates but also some real solution oscillates (flutter for example).

A specific method called the Non-Smooth Contact Dynamics (NSCD) method was developed in Montpellier by Moreau-Jean-Dubois (52) (53) for discrete problems (granular materials). A version adapted to continuous formulations and FEM was conducted at the LMA in Marseille - Jean et al (54). The system on differential measure (31) can be written in the following equivalent form: $\forall t \in [0, T]$

$$M(\dot{U}(t) - \dot{U}(0)) = \int_0^t (F - K.U - C.\dot{U})ds + \int_{[0,t]} Rd\nu \tag{38}$$

$$U(t) = U(0) + \int_0^t \dot{U}ds \tag{39}$$

where ds represents the Lebesgue measure. Be given the time discretization: $i = 0...N, ti = i.h$ (h is the time step), (38) is written:

$$M(\dot{U}(t_{i+1}) - \dot{U}(t_i)) = \int_{t_i}^{t_{i+1}} (F - K.U - C.\dot{U})ds \; + \int_{[t_i,t_{i+1}]} Rd\nu$$

and

$$\bar{R}^{i+1} = \frac{1}{h}\int_{[t_i,t_{i+1}]} Rd\nu.$$

To complete the time discretization the two following Lebesgue integrals must be approximated :

$$\int_{t_i}^{t_{i+1}} (F - K.U - C.\dot{U})ds \text{ and } \int_{t_i}^{t_{i+1}} \dot{U}ds.$$

The choice of the integration methods must be influenced by the fact that the velocity is discontinuous. We have used the following three methods:

- θ-Method : both integrals are approximated by the classical θ-method i.e. where:
  $\displaystyle\int_{t_i}^{t_i+1} fds \approx \; h(\theta f(t_{i+1}) + (1-\theta)f(t_i))$,
- θ-Euler-Method: the first integral is approximated by the θ-method and the second one by the Euler implicit method,
- modified θ-Method: both integrals are approximated by the θ-method but in the contact relations the displacement $u(t_{i+1})$ is replaced by $\hat{u}(t_{i+1}) = u(t_{i+1}) + h(1-\theta)\dot{u}(t_{i+1})$.

In contact and multibody dynamics, we have to mention the reference books by Pfeiffer-Schindler (88), Pfeiffer (87) and Pfeiffer-Glocker (86).

3.9 Numerical analysis

In the art of modeling, it is important to keep in mind the three steps between the initial mathematical problem and the final solution given by the computer code, and consequently the three levels of the analysis.

- The initial continuous problem is set in functional spaces (Hilbert spaces in our case; see Section 3.1).
- This problem is approximated by an approached problem set in finite dimensional spaces; when FEMs are used these finite dimensional spaces are generated by a basis of functions constructed on the finite element mesh; this problem is still written in functional spaces (L^2).
- The discrete problem is deduced from the previous one, which is written in $\Re^N$.

Numerical analysis is applied to the study of the properties of the solutions to these various problems and the relationship between these solutions. It is a mathematical task. An important feature is the convergence (and the order of convergence) of the approximated problem towards the initial continuous problem when the size of the refinement (element size) goes to zero. Numerical analysis is also applied to the study of the convergence of the algorithms and the evaluation of the computational errors.

Numerical analysis is of great importance in the art of modeling and constitutes a huge part of applied mathematics. It will help the engineer to choose and control the numerical methods, the algorithms and the numerical parameters.

Once again, it is not possible to be a specialist in all these topics but bearing the main numerical analysis results in mind will be very helpful to conduct a coherent modeling process.

In contact mechanics, among a number of studies, we can cite those by Glowinski- Lions-Tremolires (46) and Barbara Wohlmut (125).

3.10 Conclusions

The interaction between mechanics and mathematics is of great importance in the art of modeling mechanical systems. It concerns not only the way to set and to write the problems (choice of the formulation) but also the way to solve it. Mathematical results which could sometimes be very tough to analyze are in fact of great interest for understanding some subtleness of the models and of the solutions, and also very helpful for choosing and controlling the numerical methods.

In the light of the mathematical analysis briefly presented in this section, we can stress a few points in contact mechanics modeling:

- Contact problems are relevant to non-smooth mechanics. The non-smooth character of the basic laws (Signorini, Coulomb), beyond the simplicity of these laws, contains the ingredients for modeling many mechanical phenomena observed with frictional contact. This non-smooth character should be preserved and studied properly; regularization should be avoided as much as possible because using regularization may result in some fundamental mechanical properties being lost.
- When the friction coefficient μ is large, the problem becomes ill-posed and things take a bad turn, both for theory and for computations. Multiple solutions may occur and it is even possible to construct examples with a few degrees of freedom showing the existence of several

solutions when μ is large. The convergence of some of the algorithms is distorted when μ is large.
- Therefore, when it comes to friction modeling it is very important to keep in mind that: - when the friction coefficient μ tends to zero, the problem will tend towards a tangential free boundary condition, as intuition suggests, - but when friction μ tends to infinity, the problem does not tend towards a tangential clamped boundary condition (as intuition could suggest) but towards a very bad conditioned problem where strange phenomena may occur (multiplicity of solutions, flutter, divergence, etc.). For example, we will show in the next section that a "large μ" makes easier the occurrence of instabilities

3.11 Friction instabilities: a model to reduce squeal of a rubber-like waist seal sliding on a car window

With this example we would like to emphasize various points earlier addressed about the art of modeling:

- how much theoretical and mathematical analysis is helpful to solve industrial problems,
- making it as simple as possible when choosing the model (a constant friction coefficient is used),
- avoiding regularization in order to preserve the non-smooth character of the laws which is fundamental here (strict contact conditions are prescribed),
- using convenient tools for solving the Non-Smooth Contact Dynamic problem (the NSCD method).

The industrial problem.
The problem is to optimize the design of a waist seal in order to reduce or to avoid the squeal phenomena observed during the sliding of this waist seal on a lateral car window. Mainly, we have three constitutive characteristics to optimize: shape of the waist seal, rubber, varnish (that is, the friction coefficient). Because of the strong nonlinearities of the problem, optimization through an experimental trial/error process like the one mentioned in Section 1.1 cannot be used (too many experiments to perform).

Modeling has two goals: understanding the reasons for the occurrence of the nuisance and constructing a simulation tool to help the engineer in the design of a new seal.

Model and assumption.
Details on this study conducted within a collaboration with J.A.C. Martins at the IST of Lisbon can be found in Raous et al (98), Martins et al (72), Vola et al (123). See also the work conducted by N Guyen Quoc Son (80)

and Franck Moirot and Xavier Lorang on stability analysis with application to brake squeal.The work was done in several stages.

- First a mathematical analysis of friction stability was conducted in (72). Theorems giving the sufficient or necessary conditions for instability to occur (under a set of assumptions) in linear elasticity were established.
- Secondly, an extension of the analysis of the stability in finite deformation and nonlinear elasticity (Mooney-Rivlin) was established in (123). This was again a theoretical work. Computational tools for solving this nonlinear and non-smooth problem were developed on the basis of the NSCD method (see Section 3.8).
- The theoretical conditions for the instability to occur was interpreted on the discrete problem as generalized eigenvalue analysis on certain matrices which depend on the contact condition and have to be updated all along the solution evolution.
- Finally, the model was applied to the industrial problem.

Mathematical analysis of instability for frictional contact.
The analysis of instability for frictional problems is a difficult and still very open problem. In the present contribution, we set a basic assumption which is that there is no change in the sliding condition when instability occurs. This means that a point sliding in one direction can oscillate but without reverse sliding, only with changes in the velocity amplitude. This assumption permits a local linearization for conducting the stability analysis. This assumption seems to be very restrictive but it will be observed finally that when the dynamic solution is computed, instability occurrence satisfies the condition given by the following theorems. This suggests that these conditions could be considered quite optimal despite this initial very restrictive assumption.

The regular solutions are computed. In (72), we are interested in the initial condition problem, i.e., a Cauchy problem. In (123), we are interested in the steady sliding solution for modeling the sliding of a waist seal on a window glass. In order to study the stability of these solutions we introduce a perturbation and we evaluate the perturbed solution. The stability analysis depends on the contact status (no-contact nodes, sliding nodes, stuck nodes) either of the steady sliding solution or at every time steps of the regular solution. In every case, the possible growth of the perturbed solution is computed by considering the admissible directions depending on the contact conditions. Generalized eigenvalue problems are obtained. Details of the stability analysis can be found is the papers previously referenced. A series of theorems giving either necessary or sufficient conditions for flutter or divergence instability are then established.

Numerical analysis of the stability.
These theorems are interpreted on the discrete problem (FEM) as the analysis of generalized eigenvalue problems set on matrices which depend on the contact status of the solution (non-contact nodes, sliding nodes and stuck nodes).

The discussion is conducted both on the rate growth of the perturbed solution (real positive part of the complex eigenvalues) and on the frequency of the flutter (imaginary part of the eigenvalues).

Let us summarize the modeling process:

- let us first emphasize that we use contact and friction models without regularization and with a constant friction coefficient
- we compute the regular solutions, either Cauchy solution or steady sliding solution (small or finite deformations, linear or nonlinear elasticity, etc.)
- stability analysis is conducted on the generalized eigenvalue problems set on matrices depending on the contact condition.
- in order to evaluate the optimality of the conditions, computation of the dynamic solution using the Non Smooth Contact Dynamics method is conducted in the situation where instability is predicted to occur by the theory.

Application to the waist seal.
Two specimens of waist seals are considered (see Fig. 37). For each, the possible occurrence of squeal is studied relative to the variation of the main constitutive parameters (especially the friction coefficient). This consists in analyzing the generalized eigenvalue spectra. Validation of the model is conducted by comparing the model results and the results of experiments conducted by Renault. We will compare the two specimens presented in Fig. 37 and evaluate for each of them the possibility of squeal to occur by analyzing the eigenvalue spectrum.

First the steady sliding solution is computed for a glass window moving up and moving down for various values of the friction coefficient (finite deformations, rubber nonlinear elasticity, unilateral contact and friction) given in Fig.37. For each we need to build some matrices depending on the contact condition (see (96), (72), (123) for details). Generalized eigenvalue problems are solved. To analyze the flutter instability occurrence, the imaginary part is interpreted in terms of frequencies (see Fig. 38 and 39), the real part in terms of growth rates. The eigenmodes are also computed. A large number of results are given in (123). We only present on Fig. 38 and 39 the spectra of the flutter instability which may occur for each of the two geometries in function of the values of the friction coefficient. It can be noted that in the acoustic range (100Hz 14 000Hz), geometry 1 generates a flutter vibration (2000Hz and 14 000 Hz) for $\mu > 0.3$, although in the same range $0.3 < \mu < 0.45$ geometry 2 is stable, flutter does not occur.

Then, when the dynamics evolution of the perturbed solution is computed using the NSCD method for $\mu = 0.4$ for the two geometries, a growing oscillating solution (flutter) for geometry 1 and a stable solution for geometry 2 can be observed (see Fig. 40 and Fig. 41). This confirms that despite the very strong assumption made at the beginning of the stability analysis, the stability conditions seem to be quasi optimal.

This is validated by the experiments which show that geometry 2 is more stable (no squeal) than geometry 1 and when flutter occurs with geometry 1, the frequency of the squeal noise is about 2000Hz.

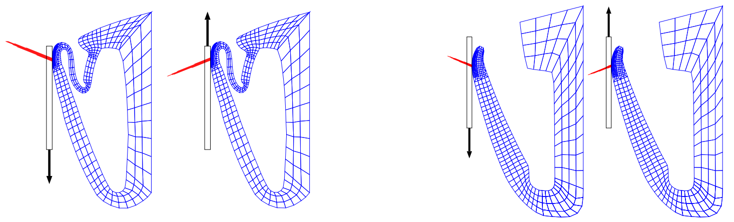


Figure 37. The two specimens: steady sliding solutions (deformation and forces) for a glass window moving up and moving down

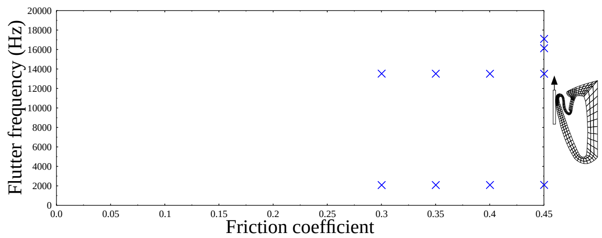


Figure 38. Flutter frequencies (geometry 1, glass moving up, $\mu = [0, 0.45]$).

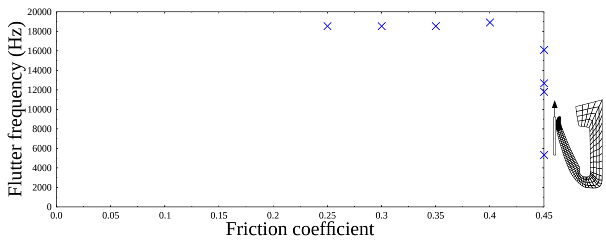


Figure 39. Flutter frequencies (geometry 2, glass moving up, $\mu = [0, 0.45]$).

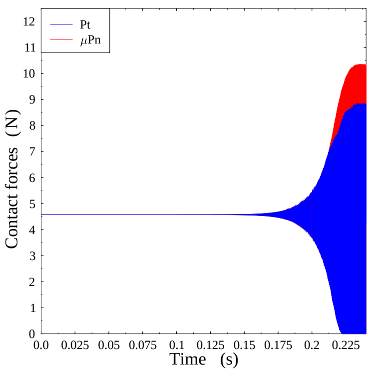


Figure 40. Evolution of the contact forces (geometry 1, glass moving down, $\mu = 0.4$, $\theta = 0.55$).

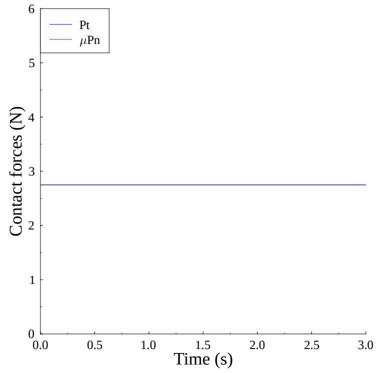


Figure 41. Evolution of the contact forces (geometry 2, glass moving down, $\mu = 0.4$, $\theta = 0.55$).

Conclusion.
With this example, we show that a mathematical analysis of stability with a model with constant friction coefficient, associated with a convenient formulation (differentiable measures for the dynamics) and with a convenient numerical method (NSCD) makes it possible to characterize friction instabilities. We show that considering a variable friction coefficient is not a necessary condition for modeling stick-slip or instability, as it has very often been asserted. We showed that the ingredients contained in the Coulomb law are sufficient to characterize this phenomenon, provided that the formulation and the numerical resolution would comply with the non-smooth character of the law.

Using this model with constant friction coefficient refers to the idea of making it as simple as possible when building a model. Introducing a variable friction coefficient would include extra parameters that are poorly known and would have disturbed the analysis.

Another important point to note is that regularizations are risky. In particular stability analysis would have been impossible if either a penalization for the frictional contact or a time regularization for the shocks had been used. One reason is that regularizations kill the oscillations and another is that the stability analysis would be completely dependent on the values of the regularization parameters!

It was necessary to preserve the non-smooth character of the contact law and to treat it correctly - which refers to the previous assertion: using the rights tools appropriate to the solution properties.

4 Identification, validation and validity domain - examples

Model validation constitutes a fundamental step in the model-building process before using the model for predicting new behaviors. Parameter identification is an intermediate step which only shows that there exists at least one set of parameters that provides a good description of a reference experiment, no more. The validation must show that the model using the previously determined constitutive parameters can give a good simulation of other experiments than the one used for the identification. Then, prediction consists in using the model in situations where no experiments have been carried out. The examples considered (geometries, load amplitude, load velocities, etc.) have to be compatible with the founding model assumptions (validity domain). As presented below, contact mechanics often requires specific procedures. This will be illustrated by several examples.

4.1 Validation of the numerical methods

The first issue is the validation of the numerical method used to solve the problem. It consists in verifying that the numerical tools have been correctly chosen and implemented, and evaluating the errors they introduce.

In contact mechanics, we need to check first if the solution fulfills the contact conditions: no penetration (or small penetration if penalization is used), only traction on the contact boundary, condition $\|R_t\|/|R_n| = \mu$ when sliding occurs, etc. Besides that, as usual, a more complete analysis has to be conducted to check the ability of the numerical method to correctly solve the problem.

The validation of a numerical method could be conducted using:

- an analytical solution when it is possible (it is the best way but is not always possible with a complex model),
- a Benchmark which is a reference example chosen by a user community which permits comparisons of the results obtained using various computational codes as well as discussions between developers. This means that a consensus has emerged within the scientific community to consider a solution as a reference solution. The characteristics of a Benchmark should be the same as those of the problem under consideration: kinematics, material behavior, load type, etc.

Analytical solutions for contact problems are rare and limited to basic cases: small deformations, elastic behaviors, simple geometries (sphere, cylinder, contact with a half-plane), simple loadings (static contact, indentation, etc.) and often frictionless contact. In contact mechanics, very often we will have to use benchmarks.

This validation of the numerical method is essential. It is absolutely required when developing a new method, and it is also recommended and very useful when using an existing method (in a commercial code for example). Furthermore it also constitutes a way to evaluate the performances of the algorithms:

- to test the convergence,
- to evaluate the errors (and to compare them with the theoretical estimates),
- to adjust the computational parameters and test the sensitivity of the solutions to their variation,
- to compare the efficiency of various methods.

Example: validation of the computational methods for solving the contact problem of a rubberlike structure with unilateral contact and friction.
An extension of the NSCD method presented in Section 3.8 was developed to compute the dynamic solution of the problem presented in Section 3.6. This new method was built to solve problems of unilateral contact with Coulomb friction (without regularization) for a rubberlike material with large deformations and non-linear elasticity (with incompressibility condition). The problem is too complex to provide a reference analytical solution which can be used to validate the method. Two Benchmarks were considered, one for quasi-static loading and one for dynamic loading.

Computation of the quasi-static compression of a cylinder.
This Benchmark was proposed and used by Simo-Taylor (113), Sussman-Bathe (117) and Liu-Hofsetter-Mang (68). Cases without friction and with friction ($\mu = 0.2$) were considered (see Fig. 42, 43 and 44)

There was very good agreement between our results and those obtained by Sussman-Bathe (117) (displacement/hydrostatic pressure formulation) and Simo-Taylor (113).

Dynamic impact of a cylinder inside a cone.
This Benchmark was proposed by Wriggers et al (126). It is the impact of a rubberlike cylinder inside a rigid cone. A good agreement was also obtained (Vola et al (123)).

With these examples, we stressed the importance of validating the numerical methods associated with a model and the difficulties finding a reference test in the case of complex problems.

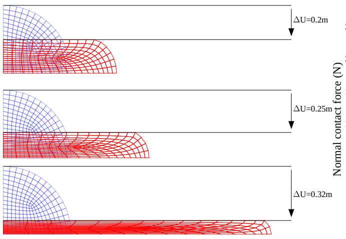


Figure 42. Deformations for the frictionless case

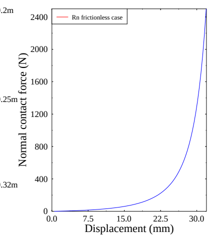


Figure 43. Resultant contact forces for μ=0

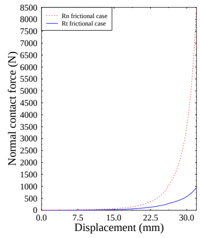


Figure 44. Resultant contact forces for μ=2

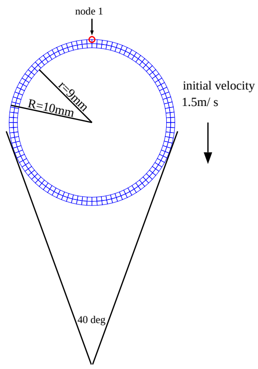


Figure 45. impact of a cylinder into an angle

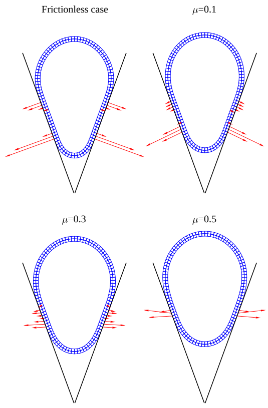


Figure 46. Deformation of the cylinder and contact forces for various friction coefficients.

4.2 Identification of the constitutive parameters.

As earlier noted, mechanics of materials and structures, it is usually possible to determine the constitutive parameters with preliminary and standard experiments conducted on specimens. However, for contact problems, the identification of the constitutive parameters (and especially the friction coefficients) is very difficult because of the strong dependence on the environmental conditions (temperature, surface condition, residual lubricant, etc.). Special methodologies are needed; for example conducting the identification on the mechanical problem itself (the complete structure) taking a given loading situation as reference. Once this is done, the validation of the model will be conducted on other cases of loadings by using the parameters determined during the identification.

It should be kept in mind that identification of the constitutive parameters is an important and difficult task. It is a key point in the art of modeling. It is a problem of minimization of the error between the experiment and model results. As said earlier, the choice of error norm is very important (L^2 or L^∞ for example). This minimization problem does not have a unique solution (a priori no convexity property of the functional to minimize). We have to deal with local minima and so the search for the correct parameters has to be enriched by estimations of ranges of values based on mechanical considerations. We will illustrate this with a few examples in Sections 4.4, 4.5 and 4.6.

4.3 Validation of the model.

Now that the numerical methods are validated and the constitutive parameters evaluated, model validation will consist in verifying whether the model using the previously determined parameters is able to describe other experiments. The experimental conditions have to satisfy the constraints created by the validity domain conditions of the model. The norm used to evaluate the error between simulation and experimental results should be the same as the one used for the identification. This will be commented and illustrated with the following examples.

4.4 Example in Mechanics of Materials: cyclic behavior of polymeric foam.

The construction of a model for the cyclic behavior of polymeric foam was presented in Section 1.9. Here, we give details about the identification step and highlight some specific ways to proceed (see Pampolini-Raous (82)). The model uses a number of parameters (to characterize viscosity, nonlinear elasticity and damage) and we will stress two points:

- the number of parameters should be minimized (as few as possible and as many as necessary): to characterize the viscosity, we will minimize the number of relaxation times to be taken into account,
- mechanical considerations should be included in selecting the parameters of the nonlinear elastic model, instead of conducting a blinded minimization process.

Identification of the viscosity parameters.
The relaxation times characterizing the viscosity were identified using extra relaxation experiments conducted on foam specimens. The relaxation curves give the evolution of the stress when an initial deformation is prescribed and kept constant. The problem is to determine the number of

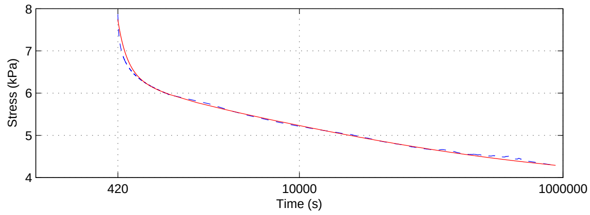


Figure 47. Relaxation experiment lasting 10 days: comparison of a model with 4 exponentials (dotted line) and 5 exponentials (full line) with the experimental curve which is indistinguishable from the full line one.

Zener/Maxwell moduli and the values of the associated constitutive parameters, which will make it possible to simulate these relaxation experiments properly. It can be shown that the problem reduces to approximate the experimental curves by a series of exponentials. The corresponding minimization problem is solved by combining an interpolation method (Prony method) and a minimization method (Hooke and Jeeves). For a model to be as simple as possible, we start with one modulus and make the identification, i.e., we determine the best values of the parameters to minimize the gap between theory and experiment. This gap remains large when only one modulus is used and so we progressively increase the number of moduli in order to take the smallest number of moduli needed to get a good approximation of the relaxation curve. This is presented in Fig. 47. Long period effects (recovery effect after a resting period) and short ones (during a cycle) coexist. Thus, very different relaxation times were identified. It turned out that five Zener/Maxwell moduli (i.e. ten parameters) were sufficient (and necessary!) to get a good description of the viscoelastic effect.

Identification of the parameters of the nonlinear elasticity.
Once the viscosity parameters were identified using the relaxation experiments, those of the nonlinear elastic springs were identified for the first loading cycle. Six parameters characterize the shape of the strain energy given in Fig. 6 : c, μ, m, β, k, a. Instead of conducting a blind optimization, we analyzed first the role of each constant in the response curve (σ_e, ϵ) of the nonlinear spring (Fig. 6 in Section 1.9). It can be seen from Fig. 48 that:

- c determines the initial slope and the value of the local maximum - Fig. 48(a);
- μ determines the second ascending branch - Fig. 48 (b);
- m determines the descending branch, the position and the value of the local maximum - Fig. 48 (c);

- when β increases, σ_e decreases when $\epsilon \simeq a$ - Fig. 48 (d);
- k determines the slope of the curve when $\epsilon \simeq a$ - Fig. 48 (e);
- a determines the position of the local minimum - Fig. 48 (f).

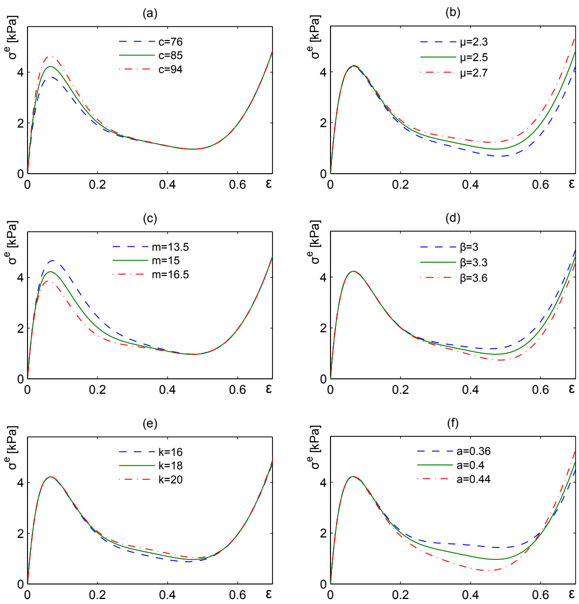


Figure 48. Nonlinear elasticity model: influence of the constitutive parameter variation on the stress/strain curve.

After this preliminary analysis, the identification procedure was performed on numerical simulations of a complete loading/unloading cycle (Fig. 49). The following procedure was used for this purpose:

1. select c to obtain the initial slope of the experimental curve;
2. select m (with c being fixed) to obtain the appropriate value of the force at the beginning of the plateau regime;
3. select μ (with c and m being fixed) to obtain the appropriate value of the force at the end of the loading process ($\epsilon = 0.7$);
4. perform an optimization routine with the Hooke and Jeeves method to determine the values of a, β and k.

The parameter characterizing the damage evolution is then identified from multi-cycle experiments. The final results are given in Fig. 49 and Fig. 50 (model results in full line; experimental ones in dotted line).

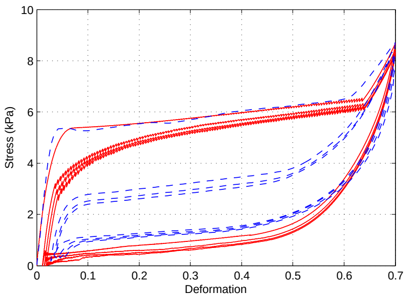


Figure 49. Theory/experiment comparison when using the nonlinear elasticity model with viscosity

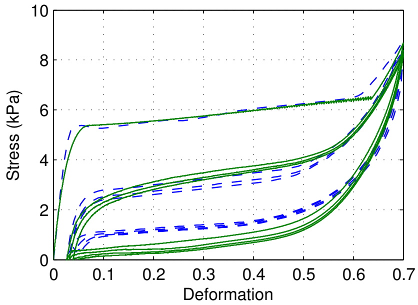


Figure 50. Theory/experiment comparison when using the model with nonlinear elasticity, viscosity and damage.

Validation of the model
In this example, model validation was achieved through simulation of complex loadings using the same values of the constitutive parameters (those resulting from the identification): cycles with various amplitudes, cycles with intermediate unloadings, etc. Results can be found in Pampolini et al (82).

4.5 Example in Mechanics of Materials : fiber/matrix interface in a composite material

With this example of micro-indentation of a single fiber in a SiC/SiC composite, which was presented in Section 2.4, we stress the importance of the mechanical analysis during the identification process.

This experiment was conducted in order to identify the constitutive parameters of the RCCM model which was developed to describe the behavior of a fiber/matrix interface. As presented in Section 2.4, the RCCM model has four parameters: the friction coefficient μ, the initial stiffness of the interface C ($C_n = C_t$), the adhesion energy ω (the Dupré energy) and the interface viscosity b.

Again, the identification procedure was not performed as a blind optimization. Preliminary studies were conducted to get a range of admissible values for each parameter. Extra experiments and energetic analysis were conducted: creep analysis (when the prescribed displacement keeps a constant value at the end of the indentation) and cyclic loading/unloading displacement of the indenter.

These preliminary studies based on mechanical considerations and some elementary computations provide:

- an estimate of the initial stiffness of the interface based on the thickness and the properties of the third body present in the interface (the fiber envelop which is a pyrolytic carbon);
- an estimate of the friction coefficient based on the analysis of dissipation during a cycle;
- a range of possible values for the adhesion energy ω based on the values of Dupré energy for these materials and energetic analysis;
- an estimate of the order of magnitude of b through experiments with different values of the indentation velocity coupled with the creep analysis.

On the basis of these preliminary estimates, the precise values of the parameters were identified using the experimental curve given in Fig. 51. Details can be found Raous-Cangémi-Cocou (95) (96).

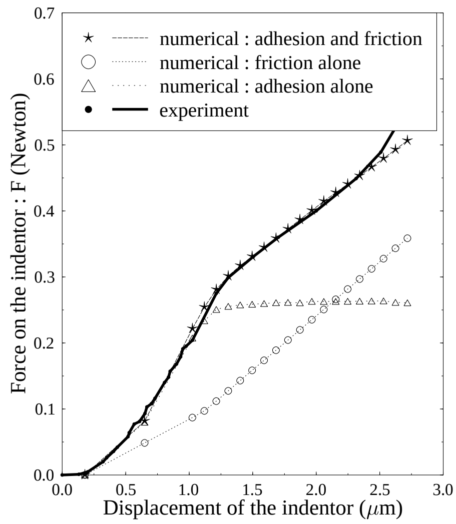


Figure 51. Identification of the constitutive parameters on an indentation experiment - Experiment (full line) - RCCM model (star line)

Validation.
Validation of the model was conducted through experiments with fibers of various diameters and different loadings.

4.6 Example in Civil Engineering: steel/concrete interface in reinforced concrete

With this example, we illustrate the ultimate step of the construction of a model (see Section 1.7): when the best simulation that can be obtained after identification of the parameters is not satisfactory, the model needs to be improved, i.e., new ingredients have to be introduced, and extra phenomena taken into account.

This is a civil engineering problem: modeling the behavior of the steel-concrete interface in reinforced concrete. The RCCM model was used to simulate the interface and an experiment consisting in pulling out a steel bar embedded in a concrete specimen was conducted (see Raous-Karray (101) (57)). The mesh is presented in Fig. 52.

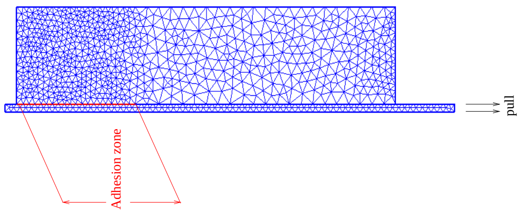


Figure 52. Pull-out of a steel bar from a concrete specimen experiment: the mesh

Identification.
During the identification process, using the standard RCCM model it was impossible to determine the values of the four parameters (μ, C, ω, b) which could give a correct simulation of the experiment: we got either a good description of the peak or a good description of the asymptote (see Fig.54). The friction coefficient μ was the key parameter. Therefore, we introduced a friction coefficient depending on the sliding displacement (see Fig. 53). From a mechanical point of view, this corresponds to taking into account a wear phenomenon corresponding to the grinding of the interface when sliding occurs, due to the powdery nature of concrete. The debris generated at the interface act as lubricant and the friction coefficient decreases when sliding occurs.

Then, it was possible to find values of the constitutive parameters C, ω, b and those defining the variation of μ given in Fig. 53 to obtain a very good approximation of the experimental result (see Fig. 55).

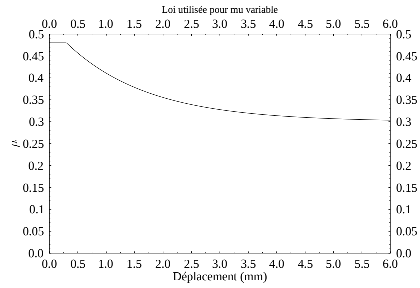


Figure 53. Variation of the friction coefficient related to the sliding displacement

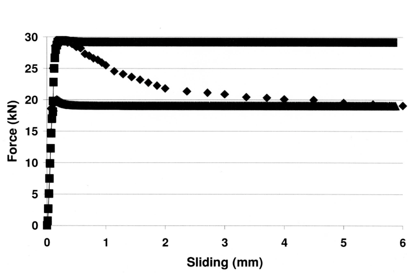


Figure 54. Approximation of the experimental curve (diamond line) by the regular RCCM model with $\mu = 0.28$ and $\mu = 0.46$.

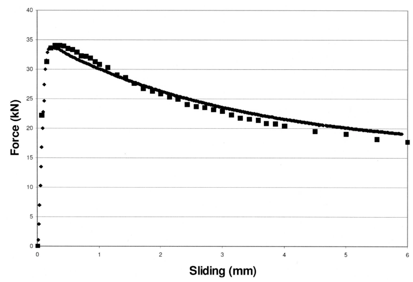


Figure 55. Approximation of the experimental curve (square line) by the model with variable friction coefficient

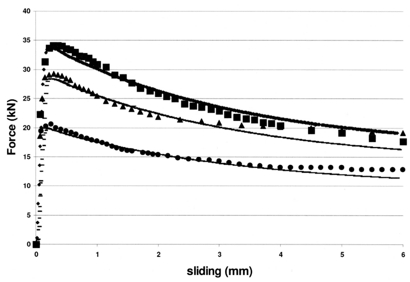


Figure 56. Simulations (full line) and experimental results for 3 rods of diameter 14mm (squares), 12mm (triangles) and 10mm (circles).

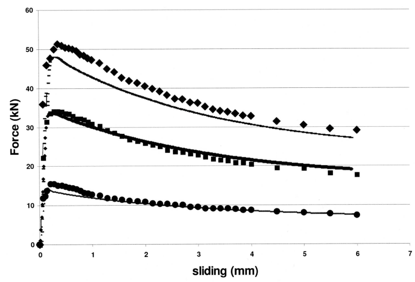


Figure 57. Simulations (full line) and experimental results for 3 lengths of the adhesive zone: 15cm (diamonds), 10cm (squares) and 5cm (circles).

Validation.
Validation of the model was performed by conducting experiments on rods of different diameters and on specimens with contact interfaces of different lengths . The results presented in Fig. 56 and 57 show the good agreement between the model simulation and the experimental results. This validates the model and in particular the choice of the function $\mu(u_t)$ given in Fig. 53.

4.7 Conclusion

Using these examples, we stressed that identification of the constitutive parameters of a model could be a difficult task (especially in contact mechanics). It is not a simple optimization problem. It is very important to take into account the mechanical meaning of the parameters as well as the range of their possible values. Complementary experiments and specific protocols may be needed (in Section 4.4, both relaxation and cyclic loadings were used and an ordered sequence was defined for the parameter determination).

In contact mechanics, identification and validation are often conducted with the same kind of experiments and special care should be brought to the processes which are conducted and to the experiments which are chosen in the two cases.

As said at the very beginning, identification of the constitutive parameters and validation of the model are two key steps to get an efficient model for simulating and predicting the behavior of a mechanical system while it is sometimes considered that the only noble part in the art and craft of modeling is the construction of the model.

What we have tried to show in this last chapter is that not only are constitutive parameter identification and model validation two key points in the art of modeling, they also are real scientific approaches.

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